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SHELDONS'

COMPLETE ALGEBRA.

PART I. BEING SHELDONS' ELEMENTS OF ALGEBRA;

PART II. BEING MORE ADVANCED ALGEBRA.

SHELDON & COMPANY,
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PUBLISHERS' PREFACE.

PART I. of this book contains a treatment of all the subjects usually found in an elementary algebra. Lengthy and unnecessary explanations, and side issues of every kind, have been carefully excluded. For the reason that at the student's first acquaintance with algebra he finds it difficult to understand general demonstrations, they have for the most part been omitted. Instead of such demonstrations, the analogies between the processes of arithmetic and those of algebra have been brought out by means of numerical illustrations.

The aim has been to furnish an abundance of exercises properly graded, and sufficiently easy for the pupil of average ability.

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A want long felt by teachers of algebra is a book containing examples, the solution of which shall really discipline the student in the principles of algebra, and give him dexterity in performing its operations.

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Albert Carlton,
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Again, if a man has $50 and pays $30 of it for a coat, actual subtraction gives $20 as what remains.

But if he has m dollars and pays n dollars for a coat, what remains is expressed by writing the sign — after m, the greater, and before n, the less; thus, $m - n$. It is read, *m minus n.*

5. The **sign** —, called *minus,* denotes **subtraction.**

EXERCISES.

6. *1.* Indicate the sum of a and b. The difference between a and b.

2. Indicate the sum of x and y. The difference between x and y. The sum of m, n, and r. Of a, b, and c.

3. Indicate the sum of p and q diminished by t. The difference between a and b increased by c.

4. I had a dollars, earned b dollars, and then spent c dollars for a coat. How much had I left?

5. A is a years old and his wife b years old; express the sum of their ages. Express the difference between their ages, if A is older than his wife. Express the difference if the wife is the older.

6. A man paid c dollars for a horse and e dollars for a carriage; what did both cost?

7. Leaving home, I rode in the cars m miles on Monday and n miles on Tuesday; how far was I then from home? On Wednesday I rode back r miles over the same road; express my distance then from home.

8. A boy received a cents from his father, b cents from his mother, and spent c cents of what he then had; express what he had left.

9. A boatman rowed p miles up a river, was towed q miles farther by a steamer, and then floated down stream 12 miles; how far was he from the place of starting?

10. A man bought a house for b dollars and sold it so as to gain c dollars; what did he get for it?

11. A merchant bought goods for t dollars and sold them so as to lose u dollars; what did he get for them?

12. I have c dollars; I owe A d dollars and B e dollars; how much will be left after I pay them?

7. Let the pupil read the following algebraic expressions, and illustrate them by means of examples similar to the foregoing:

1. $a + b$.	*7.* $x + y + z$.	*13.* $a + b - c$.
2. $x + y$.	*8.* $a + c + 6$.	*14.* $a - b + c$.
3. $m + n$.	*9.* $a - c$.	*15.* $m - n - r$.
4. $a + 8$.	*10.* $m - n$.	*16.* $a - c + 8$.
5. $a + b + c$.	*11.* $p - q$.	*17.* $x - y + z$.
6. $m + n + r$.	*12.* $a - 4$.	*18.* $p + q - 20$.

8. The product of two or more numbers represented by figures is indicated by the sign of multiplication.

Thus, 6×5; $4 \times 3 \times 6$.

9. The product of two or more numbers represented by letters is indicated by writing the letters together with no intervening sign.

Thus, ab expresses the product of a and b; mnr expresses the continued product of m, n, and r.

Hence, at a cents each, b oranges will cost ab cents.

Again, at m cents an hour, a man working n hours a day for r days will earn in all mnr cents.

10. A number written before a letter denotes the product of the letter by the number. Written before the product of two or more letters, it denotes that the product of the letters is to be multiplied by the number.

Thus, if a man earns a dollars per day, in 5 days he will earn $5a$ dollars.

Again, 5 casks of syrup each containing a gallons will cost, at b cents a gallon, $5ab$ cents.

11. A **co-efficient** is a number written before a letter, or before the indicated product of two or more letters. It shows how many times the letter or product is taken.

When no number is written, 1 is understood.

Thus, 5 is the co-efficient of $5x$, $5ax$, $5abc$, etc.

1 " " " x, ax, abc, etc.

NOTE.—Strictly, one or more of the factors of a product may be regarded as the co-efficient of the remaining factors.

Thus, in the product $3abx$,

3 is the co-efficient of abx,
$3a$ " " " bx,
$3ab$ " " " x.

EXERCISES.

12. *1.* Express the product of x and y. The continued product of x, y, and z.

2. Express the product of mn and 8. Of 2 and xyz.

3. Express the value of 8 loads of potatoes, each containing m bushels, worth x dimes a bushel.

4. A grocer bought 15 tubs of butter, of p pounds each, at q cents a pound; how much did the butter cost him? If he sold the butter so as to gain r cents a pound, what was his entire gain?

5. A boy bought a apples at m cents each, and sold them at n cents each. If he gained, what was his gain?

6. A boy worked a hours a day for 6 days at b cents an hour. With the money he bought a hat for c cents; how much money was left?

7. A ferry-boat carried an average of p passengers daily for d days, the fare being f cents for each passenger; what were its total earnings?

8. A grocer bought a crates of peaches at b cents a crate, and sold them at c cents a crate. If c is greater than b, did he gain or lose? How much?

9. A man bought a barrels of apples at b cents a bushel, and c barrels of onions at d cents a bushel. What did he pay for both, if the barrels held m bushels each?

13. ***Read the following algebraic expressions, and illustrate them by means of examples similar to the foregoing:***

1. $15ab$.	*5.* $amn - amr$.	*9.* $ab + cd + mn$.
2. abm.	*6.* $5ab + d$.	*10.* $abc + 2bd - m$.
3. $12xy$.	*7.* $3xy - p$.	*11.* $axy + bmy - 8$.
4. $ab - ac$.	*8.* $abc - abd$.	*12.* $pq + pmr + mnr$.

14. Division in algebra is generally indicated by writing the numerator above, and the denominator below, a horizontal line. Thus, $\frac{a}{b}$ denotes the division of a by b; $\frac{3x}{4}$ or $\frac{3}{4}x$ denotes the division of $3x$ by 4, or what is the same, three fourths of x.

In like manner, $\frac{a+b}{x-y}$ denotes the division of the sum of a and b by the difference between x and y.

EXERCISES.

15. *1.* Write the product of x and y divided by their sum. Their product divided by their difference.

2. Express 5 times the product of a, b, and c divided by the difference between m and n. Express the same product divided by the sum of m and n.

3. At the rate of 5 miles an hour, how long will it take to go 15 miles? How long to go a miles? To go am miles? To go $3ab$ miles?

4. A man paid a dollars for wood at b dollars a cord; how many cords did he buy?

5. I walked m miles at the rate of r miles an hour; how long did it take me?

6. A grocer bought a barrels of potatoes for c dollars; how much was that a barrel?

7. Mr. Brown drove 2 hours at the rate of 10 miles an hour, and walked back at the rate of 4 miles an hour; how long did it take him?

8. A man drives for m hours at the rate of p miles per hour; how long will it take him to walk back at the rate of r miles an hour?

9. A merchant bought b yards of cloth at c dollars a yard, and sold it all at a gain of d dollars a yard; how much did he get for it all? How much for each yard?

10. A speculator bought h horses at d dollars each, and sold them at a loss of l dollars on each horse; what did he get for them?

11. A man traveled on the first of two days n hours, at the rate of r miles an hour; on the second day he traveled p hours, at q miles an hour. How long would it take him to return over the entire distance, at the rate of t miles an hour?

12. If I pay a dollars for b tons of coal, how much at that rate will c tons cost?

13. If c cords of wood cost \$$d$, what will e cords cost?

16. The **sign of equality**, $=$, is used in arithmetic to indicate that the numerical expressions between which it stands are equal.

Thus, $2 + 4 = 6$; $3 \times 2 \times 4 = 24$.

17. In like manner, the sign of equality is employed in algebra to show that numbers, represented either wholly or in part by letters, are equal.

Thus, $a + b = 10$ denotes that a and b represent two numbers whose sum is 10.

$a + b = mn$ denotes that a and b represent numbers such that their sum is equal to the product of the numbers represented by m and n.

$am + 5 = pq - 7$ denotes that the product of a and m increased by 5 is equal to p times q diminished by 7.

18. An **equation** is an expression of the equality of two quantities or sets of quantities.

19. The **first** or **left member** of the equation is the part that precedes the sign of equality.

The **second** or **right member** of the equation is the part that follows the sign of equality.

Thus, of the equation $5a - b = 2x + y$, the first or left member is $5a - b$, and the second or right member is $2x + y$.

20. An **identical equation** is one whose members have the same form, or are reducible to the same form.

Thus, $2 \times 3 = 6$, and $ax + y = ax + y$, are identical equations.

EXERCISES.

21. *1.* Indicate that the sum of a and b is equal to m times n.

2. Indicate that 4 times a is equal to b diminished by c.

3. Write an equation showing that the sum of a, b, and c is equal to 20 diminished by m.

4. Indicate that the sum of m and n diminished by d is equal to the product of a, b, and c.

5. Indicate that the product of a and b increased by the product of c and d is equal to 3 times the product of m and n.

6. The letters a, b, c, and d stand for numbers such that the product of a and b divided by the sum of a and b is equal to the difference between c and d divided by the product of c and d. Write an equation showing this.

7. With $\$a$ I paid for 5 bushels of potatoes at $\$b$ a bushel, and received $\$c$ change. Express this as an equation.

8. Indicate by an equation that a lb. of beef, at b cents a lb., cost c cents more than m lb. of mutton at n cents a lb.

9. Express by an equation the fact that a coat for a dollars and m yards of cloth at n dollars a yard cost together the same amount as b cords of wood at c dollars a cord.

10. A can travel m miles at the rate of n miles an hour in the same time that B can travel a miles at c miles an hour. Show this by an equation.

11. A grocer buys a pounds of butter at b cents a pound; he sells it at d cents a pound and gains by the transaction e cents. Express the fact by an equation.

12. A father works a weeks at b dollars a week and his son works m weeks at r dollars a week. With the money they pay for t tons of coal at s dollars a ton and have n dollars left. Write the equation.

22. The **signs of aggregation** are the *parenthesis*, ( ), the *bracket*, [ ], the *brace*, { }, the *vinculum*, $\overline{\quad}$, and the *bar*, |. Each denotes that the several quantities to which it is affixed are to be regarded as a single quantity.

Thus, $(x + y)$, $[x + y]$, $\{x + y\}$, $\overline{x + y}$, and $\begin{matrix} x \\ + y \end{matrix}\Big|$ indicate that $x + y$ is to be treated as if it were represented by one letter.

23. A quantity in parentheses may have a co-efficient to show how many times it is to be taken.

Thus, $5\,(a + b)$ denotes that the sum of a and b is to be multiplied by 5.

$ab\,(m - n)$ denotes that the difference between m and n is to be multiplied by the product of a by b.

24. When two or more parenthetical expressions are written together without intervening signs, their product is denoted.

Thus, $(a + b)(a - b)$ denotes the product of the sum of a and b by their difference.

$3m\,(a + m)(b - n)$ denotes $3m$ times the product of $(a + m)$ by $(b - n)$.

25. Two or more of the signs of aggregation may be used together.

Thus, $3\{12 - (5 + 3)\}$ denotes that $5 + 3$ considered as a single quantity is to be taken from 12, and the remainder multiplied by 3. The value of the expression is therefore equal to 3×4, or 12.

$20 - \{12 - [5 + (7 - 4)]\}$ denotes that $7 - 4$, or 3, is to be added to 5; that the sum is to be taken from 12; and that the remainder, 4, is to be taken from 20, giving 16.

EXERCISES.

26. *1.* What is the value of $5(2+3)$? Of $(2+3)(5-2)$?

2. Express 3 times the sum of m and n; 5 times the difference, m being greater than n.

3. A boy had 25 cents; he paid 12 cents for a ball and 5 cents for a top; indicate by a parenthesis what was left.

4. A man had a dollars; he paid his grocer b dollars and his butcher c dollars; indicate the remainder by a parenthesis.

5. A grocer had 30 bushels of potatoes in one bin and 20 bushels in another; of these he sold to one customer 12 bushels and to another 10 bushels. Express the remainder by means of parentheses.

6. A farmer sold some corn for c dollars and some potatoes for p dollars; of the money he paid h dollars for a hat and m dollars for a muff for his wife. Show by parentheses what remained.

7. A man had m children; he bought for each a pair of shoes at a dollars a pair and a suit of clothes at b dollars. Indicate the entire expenditure.

8. Show that the following equations are correct:

$$3[9-(7-5)] = 21.$$

$$(4+5)(12-3\times 3) = 27.$$

$$(2+3)(4+2)(9-7) = 60.$$

9. What is the value of $20-[18-(15-7)]$? Of $5(15-\overline{5+6})$? Of $(5+3)(8-2\times 3)$?

27. In the notation of arithmetic, when a number is to be taken in a product two or more times as a factor, there are two ways of indicating it.

Thus, 5×5, or 5^2, expresses the multiplication of 5 by itself.

28. In the notation of algebra, when a letter is to be taken in a product two or more times as a factor, it is written but once, and the number of times it is to be taken as a factor is shown by a small figure called an **exponent.** When no exponent is written, 1 is understood.

Thus, $3aabbbc$ is written more briefly $3a^2b^3c$.

29. When a quantity enclosed in parentheses is to be taken in a product more than once as a factor, an exponent is employed to indicate how many times it is taken.

Thus, $(a + b)^2$ means $(a + b)(a + b)$, and is read, *the square of* $a + b$.

$3a^2b(m - n)^3$ means $3a^2b(m - n)(m - n)(m - n)$, and is read, *three a square b times the cube of* $m - n$.

30. A product denoted by an exponent is called a **power.**

Thus, a^2, $(a + b)^2$, $(mn)^2$, $(ab + 1)^2$, are second powers, or squares.

x^3, $(x - y)^3$, $(5a)^3$, $(xy)^3$, are third powers, or cubes.

EXERCISES.

31. ***Read:***

1. a^3, a^2x, $5a^2b^3c^3$, $13a^5x^8$, $x^2b^4xy^3$, $7r^4y^7z^3$.

2. $(a + b)^2$, $(a - b)^3$, $a^2 - b^2$, $x^2 + 2xy + y^2$, $(a^2 - b)^2$, $a^2b^2 - 2ab^2xy + x^2y^2$.

3. $5a^2(x + y)^2$, $(a + b)^5$, $a + b^3$, $(a + b)^3$, $6x^2(m - n)^2$.

4. $(5a - b)(x - 3y)^2$, $(a^2 - b^2)(a^2 + b^2)^2$, $(ab - c^3)^2$, $15a(x - y)^2(x + y)^3$.

Indicate:

5. The square of the sum of x and y. The square of the difference between x and y.

6. Three times the square of the sum of a and b. Five a times the cube of the difference between a and b.

7. The product of $5a$ by the sum of x square and y.

8. The product of x square minus y square, by x square plus y square.

9. The continued product of the cube of x, the square of y, and the fourth power of z.

10. The square of the sum of a square and b. The square of the sum of a and b square.

32. An expression like $5ab$ is called a **term**; the coefficient 5 is called the **numerical part**; ab is called the **literal part**; a and b separately are **literal factors**.

33. Similar terms are terms having the same letters affected with the same exponents.

Thus, $6a^2b^3c$, $3a^2b^3c$, $-5a^2b^3c$ are similar terms.

34. Dissimilar terms are terms that differ in their literal factors or in their exponents.

Thus, $5a^2bc$, $3ab^2c^3$, $-6ab^2m^3$ are dissimilar terms.

35. A **monomial** is an algebraic expression consisting of one term.

Thus, $6x$ and $15ab^2c$ are monomials.

36. A **polynomial** is an expression consisting of two or more terms.

Thus, $x + y$ and $a + b + x^2y - 6z$ are polynomials.

37. A **binomial** is a polynomial of two terms.

Thus, $x + y$ is a binomial.

38. A **trinomial** is a polynomial of three terms.

Thus, $ab + cd - x$ is a trinomial.

39. The **numerical value** of an expression is found by substituting for the letters contained in the expression their values, and performing the operations indicated by the signs.

Thus, if in the following expressions,

a stands for 2,
b " " 3,
and c " " 4,

then
$$abc = 2 \times 3 \times 4 = 24,$$
$$ab\,(b + c) = 2 \times 3\,(3 + 4) = 6 \times 7 = 42,$$
$$(a + b)\,(b + c) = (2 + 3)\,(3 + 4) = 5 \times 7 = 35,$$
$$a^2 + b^2 = 2^2 + 3^2 = 4 + 9 = 13,$$
$$\frac{abc}{a + c} = \frac{2 \times 3 \times 4}{2 + 4} = \frac{24}{6} = 4.$$

EXERCISES.

40. Find the numerical value of the following expressions when $a = 3$, $b = 5$, $c = 8$, $x = 10$, $y = 4$, $z = 6$, $u = 9$.

1. ab.	*5.* b^2xy.	*9.* $xy\,(xy - y^2)$.
2. a^2bx.	*6.* $2ab\,(a + b)$.	*10.* $uz\,(z + u)$.
3. ac^2y.	*7.* $3x^2\,(x - y)$.	*11.* $(x - y)^2$.
4. a^3bz.	*8.* $a\,(x^2 + u^2)$.	*12.* $(b^2 - x)^2$.

13. $(u - z)^3$.

14. $b(u^2 - c^2)$.

15. $ab(u - c)^2$.

16. $3a(x + y)$.

17. $a^2 + bc$.

18. $y^3z + 3zu$.

19. $5abc - xyz$.

20. $x^2y - xy^2$.

21. $ab + x - yz + u$.

22. $b(a + b) \div (y + z)$.

23. $(x + y)(x - y)$.

24. $(a^2 - c)(u^2 + x^2)$.

25. $a^2 + b^2 - (x^2 - u^2)$.

26. $(z^2 - y^2) - (b^2 - a^2)$.

27. $x^2 - \{u^2 - (c^2 - a^2)\}$.

28. $bc - (x + \overline{u^2 - c^2})$.

29. $\left(\frac{ax}{b} + z\right) \div y$.

30. $b + \frac{c - a}{4} + u$.

31. $\frac{ab + c - (a + x)}{b}$.

32. $\frac{2ab}{y} + \frac{2xy}{c} + \frac{b + y}{u}$.

33. $3x\left\{\left(\frac{u^2}{a + z} - \frac{a + b}{c}\right) \div c\right\}$.

34. $\frac{c^2 + x - b}{a} + \frac{(z^2 + u)x}{b}$.

35. $\frac{a + b^2 + c + (x^2 - c^2)}{u^2 - (c^2 + u)}$.

36. $\frac{(a + b)(b + c)(x - c)}{a^2 + y}$.

37. $\frac{5(xyz - abc)}{bc - ax} + \frac{a^2 + b^2 + c^2}{3ab + y}$.

38. $3a^2x^2y - \{2a^3xz - [(a + b)(y^3 - u)]\}$.

41. In every arithmetical problem certain numbers are given under such conditions that other numbers not given may be found.

Thus, if it is known that a farmer has \$20 left after paying for 30 sheep at \$4 each, the money he had at first may be determined.

In this example there are four elements, viz. :

Three **known,**
1. *Price of one sheep.*
2. *No. of sheep bought.*
3. *Money left after the purchase.*

One **unknown,** that may be found, *The money at first.*

42. In the application of algebra to the solution of problems, **known quantities,** or quantities that are assumed as known, are represented by the *first letters of the alphabet,* or by those letters in combination with figures, or by figures alone ; as, a, b, $3a$, etc.

43. Unknown quantities, or those whose values are not given, are represented by the *last letters of the alphabet,* as, x, y, z, etc.

44. In the solution of problems in algebra, the *equation* is always employed.

In arithmetic, the known quantities are combined in accordance with the conditions imposed, and the unknown quantity is finally determined.

In algebra, the unknown quantities are represented by letters, and then, as if they were known, they are combined with the known quantities in the form of an equation. The solution of the problem then consists in determining what values those letters must have in order that the members of the equation may be equal to each other.

A solution in algebra therefore resembles what is called the proof in arithmetic. This may be seen by an inspection of the following :

From New York to Philadelphia is 90 miles. In 3 days a traveler accomplished all but 15 miles of the distance. What was his rate per day?

Solution by Arithmetic.

90 miles, entire distance.
15 " , what he lacked.
3) 75 " , what he traveled in 3 days.
25 " , " " " " 1 day.

PROOF. $(3 \times 25) + 15 = 90$.

Solution by Algebra.

Let $x =$ his daily rate,

then $3x =$ distance traveled in 3 days.

Hence, $3x + 15 = 90$.

Since 15 must be added to $3x$ to make 90, $3x$ is 15 less than 90, equal to $90 - 15$, or 75.

If $3x = 75$,

$x = 75 \div 3$, or 25, his daily rate.

45. Algebra is that branch of mathematics in which the relations of quantities represented by letters and figures are investigated by means of the equation.

The practical utility of algebra consists in the application of the equation to the solution of problems.

46. In order that the pupil may be able to employ the equation in its more difficult forms in solving problems, much practice is necessary in adding, subtracting, multiplying, dividing, and factoring algebraic expressions. Before enter-

ing upon that practice, however, the following simple exercises are given for the purpose of indicating more clearly the nature and use of the equation.

EXERCISES.

47. *1.* If x is equal to 5, what is the value of $2x$? Of $3x$? Of $4x$?

2. If $3x$ is equal to 15, what is the value of x? Of $2x$? Of $5x$?

3. If x increased by 9 equals 19, what is the value of x?

4. If y diminished by 9 is equal to 10, what is the value of y?

5. The value of $2x$ increased by 5 is 25; what is the value of $2x$? Of x?

6. When $3x + 2 = 14$, what is the value of x?

7. Find the value of y when $3y + 2 = 20$. When $2y + 11 = 31$.

8. Find the value of x in the following: $4x - 4 = 20$; $3x + 7 = 28$; $2x - 13 = 23$; $7x - 11 = 38$.

9. What is the value of x when $x + 2x = 21$? When $2x + 5x = 21$? When $x + 2x + 4 = 28$? When $x + 2x - 5 = 25$?

10. A coat and a hat cost together \$25; what was the cost of each, if the coat cost 4 times as much as the hat?

Solution.

Let $x =$ the cost of the hat.

Then will $4x =$ " " " " coat;

and $x + 4x =$ " " " both.

Hence, $x + 4x = \$25$;

or $5x = \$25$;

whence, $x = \$25 \div 5 = \$5$, the cost of the hat,

and $4x = \$5 \times 4$, or \$20, " " " coat.

11. A farmer sold some eggs and 5 pounds of butter for 95 cents; what did he get a pound for the butter if he received 35 cents for the eggs?

12. To pay a bill of $4, a man gave 6 bushels of potatoes, and by so doing overpaid the bill by 20 cents; what was he allowed per bushel for the potatoes?

13. The cost of 6 tons of coal and some wood was $23; what was the coal a ton if the wood alone cost $5?

14. A ball cost 20 cents more than twice as much as a bat, and together they cost 65 cents; what was the cost of each?

15. Mary picked some berries, and Jane picked 7 quarts less than twice the quantity that Mary picked. Together they picked 29 quarts. What quantity did each pick?

16. Mr. Brown is 12 years less than 5 times as old as his son, and the sum of their ages is 60 years. How old is each?

17. A boy paid 15 cents more than 3 times as much for a book as for a slate, and together they cost him $1.35. Find the cost of each.

18. Ann bought 12 yards of ribbon, and Kate bought 7 yards of the same kind of ribbon. Ann's ribbon cost 70 cents more than Kate's. What was the price per yard?

19. A drover bought two horses, paying the same sum for each. One he sold at a gain of $20, the other at a gain of $35. For both he received $415. What did he pay for them?

20. I bought two houses, paying $500 more than twice as much for the first as for the second. I sold both for $7900, gaining $800 on the first and losing $900 on the second. What did I pay and what did I receive for each house?

ADDITION.

48. The **sum** of two or more algebraic quantities is the simplest expression for their aggregate value.

49. **Addition** is the process of finding the sum of two or more quantities.

50. A **positive** quantity is one that has the sign + before it, either expressed or understood. When no sign is expressed, + is understood.

51. A **negative** quantity is one that has the sign — before it.

52. In arithmetic the signs + and — are always symbols of operation, but in algebra they are employed sometimes as symbols of **operation** and sometimes as symbols of **quality.** The graduation of a thermometer conveniently illustrates this extension of meaning. When the mercury stands at any distance, say 10°, above a fixed point called zero, the temperature is said to be + 10°; an equal distance below zero is — 10°. The symbol + here does not mean that 10 is to be added, nor does — mean that 10 is to be subtracted. They are correlative in meaning; they simply indicate opposite *quality* or *character.*

If motion or position *north, east, up stream, above zero, to the right, forward, up hill,* etc., is expressed by +, motion or position *south, west, down stream, below zero, to the left, backward, down hill,* etc., is expressed by —.

ORAL EXERCISES.

53. *1.* A rows 10 miles up a river from a given point, and B rows 8 miles down the river from the same point; how may the position of each with respect to the point of starting be represented?

2. The town of A is in 35° north latitude and the town B is in 20° south latitude; represent algebraically the position of each with respect to the equator.

3. A miner works in a mine a feet below the level of the sea, and lives in a house b feet above the level of the sea. Express algebraically the two locations, as compared with the level of the sea.

4. A man has $\$12p$ dollars in bank; he draws out $\$3q$, deposits $\$5m$, draws out $\$9n$, deposits $\$6r$; what is his bank account?

5. In a series of transactions a merchant's gains and losses were as follows: $\$a$ gain, $\$3b$ loss, $\$5c$ gain, $\$2m$ loss, $\$4n$ loss, and $\$3r$ gain. Write the algebraic expression showing the result of the transaction.

6. How may the financial condition of a man be expressed algebraically, who, when all his debts are paid, has property worth $\$m$, and $\$n$ in cash? How, if he has property worth $\$d$ and *owes* $\$e$?

7. An army advances m miles and retreats r miles; what algebraic expression represents its position with respect to the point of starting? What, if it advances $3a$ miles, retreats $2b$ miles, and then advances $5c$ miles?

8. The mercury in a thermometer stands at noon on Monday $5a$ degrees above zero, by noon on Tuesday it has risen b degrees, and by noon on Wednesday it has risen $2c$ degrees more; at what point does it then stand?

9. A lives m miles north of a certain town, and B n miles south of the same town. Express these locations algebraically with respect to the town.

10. If m in Ex. 9 is 20 miles and n 30 miles, express algebraically the location of A's residence *with respect to* B's. B's with respect to A's.

11. A is worth nothing, and owes B $5a$ dollars, C $7b$ dollars, and D $9c$ dollars. Express algebraically his financial condition.

12. Mr. A owes his grocer $\$3t$, he buys goods to the amount of $\$2q$, he pays on account $\$3m$, buys to the amount of $\$5n$; what does he still owe?

13. A man rows a boat $3c$ miles up stream, drifts $2d$ miles down stream, ascends $4m$ miles, and then rows down $5n$ miles; how far is he from the starting-point, and in what direction?

14. By steam alone a certain vessel moves at the rate of m miles an hour, and by sails alone at n miles an hour. At what rate per hour would the vessel under both steam and sail ascend a river the current of which is r miles per hour? At what rate would it descend the river?

ADDITION OF MONOMIALS.

54. To add similar monomials with like signs.

1. Required to find the sum of $6a$ and $3a$; also of $-6a$ and $-3a$.

Explanation.—Since 6 units of any kind and 3 units of the same kind are together equal to 9 units of that kind, it is obvious that 6 times the quantity expressed by $+a$ and 3 times the quantity expressed by $+a$ are together equal to 9 times the quantity expressed by $+a$, equal to $+9a$. In like manner, 6 times $-a$ and 3 times $-a$ are 9 times $-a$, equal to $-9a$.

Process.

$+6a$	$-6a$
$+3a$	$-3a$
$+9a$	$-9a$

55. To add similar monomials with unlike signs.

2. Required to find the sum of $5a$, $-8a$, $9a$, $2a$, and $-3a$.

Explanation.—The sum of the positive quantities is equal to $7a + 9a = 16a$. The sum of the negative quantities is $(-8a) + (-3a) = -11a$. Since $16a = 11a + 5a$, the quantity $-11a$, being of opposite quality from $+11a$, will destroy it; in other words, the sum of $+11a$ and $-11a$ is 0, and the sum of $+11a$ and $-11a$ and $+5a$ is equal to $+5a$.

Process.

$$\begin{array}{r} +7a \\ -8a \\ +9a \\ -3a \\ \hline +5a \end{array}$$

56. To add dissimilar monomials.

3. Required to find the sum of $6a$, $-3b$, $-4c$, and $6m$.

Explanation.—Since the monomials are dissimilar in form, their addition can only be indicated. This is done by connecting them with the signs that indicate their *quality*.

Process.

$$6a + (-3b) + (-4c) + 6m = 6a - 3b - 4c + 6m.$$

To illustrate, suppose that a man having \$$6a$, spends \$$3b$, and afterward \$$4c$, and finally earns \$$6m$. The foregoing expression will show the aggregate of his various operations.

57. To add polynomials.

4. Required to find the sum of $2a + 4b - c$, $6a - 2b + 5c$, $9b - 4c + 5y$, $-7a + 3c - 6y + 3m$.

Explanation.—The polynomials are written so that similar terms shall stand in the same column. The algebraic sum of each column is then found in the manner already explained.

Process.

$$\begin{array}{rrrrr} 2a & + & 4b & - & c \\ 6a & - & 2b & + & 5c \\ & & 9b & - & 4c + 5y \\ -7a & & & + & 3c - 6y + 3m \\ \hline a & + & 11b & + & 3c - y + 3m \end{array}$$

58. Rule.—I. For Similar Monomials.—*If the quantities have the same sign, prefix the common sign to the sum of the co-efficients and annex the literal part.*

If the quantities have different signs, add the co-efficients of the positive quantities and the co-efficients of the negative quantities, to the difference between these sums prefix the sign of the greater and annex the literal part.

II. For Dissimilar Monomials.—*Write them in succession with positive terms preceded by + and negative terms by —.*

III. For Polynomials.—*Write the expressions so that similar terms shall fall in the same column. Combine each group of similar terms into one term, and write the result underneath with its own sign.*

EXERCISES.

59. ***Find the sum of:***

1. $3a$, $2a$, a, and $4a$.
2. $5x$, $6x$, $3x$, and $2x$.
3. $2b$, $-3b$, $-b$, and $14b$.
4. $5ax$, $6ax$, $-9ax$, $3ax$, and $-10ax$.
5. $-6abc$, $4abc$, abc, $-2abc$, and $5abc$.
6. $2y$, $-5y$, $6y$, $3y$, and $-y$.
7. $2(a+b)$, $3(a+b)$, and $5(a+b)$.
8. $2(b+c)$, $-4(b+c)$, and $10(b+c)$.
9. $5(x+y)$, $7(x+y)$, and $a(x+y)$.
10. $x(a+b)$, $y(a+b)$, and $z(a+b)$.

11. $2x - y$, $3x + 5y$, $-7x + 3y$, and $3x - 4y$.

12. $3x + 2y + 5z$, $4x - 7y + 8z$, and $-5x + 3y - 10z$.

13. $ax - 4ab + bd$, $3bd - 2ax + ab$, $7ab - 2ax - bd$, and $5ab - 3ax + 12bd$.

14. $3abd + 4abx - 5cx$, $8cx - 11abx + 12abd$, $9abx - 12cx + 3abd$, and $7cx - 15abx + 3abd$.

15. $7b - 2a^2 - xy$, $5a^2 - 6b + 3xy$, $b - 3a^2 + 4$, and $a^2 - b - 3xy$.

16. $3b^2 - 2a^3 + 13$, $3a^3 - 2b^2 - 5$, $4ab + 7a^3 - 3$, and $2b^2 - a^3 + ab$.

17. $ax^2 - 2y + b$, $2y + 2ax^2 - 3b$, $4ax^2 - y + b$, and $2b - 3ax^2 + b$.

18. $2c^2 + a^2 + 3bc$, $5c^2 - 3a^2 - 2bc$, $c^2 + 2a^2 - bc$, and $b + bc - 3$.

19. $ab + a^2c - 5$, $3ab - 3a^2c + 7$, $2a^2c - 2ab - 3$, and $ab + a^2c + 5$.

20. $3b^2 - 2a^2x + b$, $-b^2 + 3a^2x - 3b$, $b^2 - a^2x + c$, and $3b^2 + b - 3c$.

21. $2a^2 + 3 - ac$, $3a^2 - 7 + ac$, $3ac - 5a^2 + 9$, and $3ac + 4 - a^2$.

22. $b^2c + 2 - y^2$, $y^3 - 3b^2c - 10$, $2b^2c - 3 + 2y^2$, and $b^2 - y^2 + 5$.

23. $5a + 3b - \frac{1}{2}c + d$, $2b + c - 3d$, and $7a - 5b + \frac{1}{4}c + 2d$.

24. $x^2 + \frac{1}{4}x - 9$, $\frac{3}{4}x + 17 + y$, $3x^2 - x - y$, and $-8 + y^2$.

25. $8m + 9n + x - y$, $\frac{1}{2}m - 2n + 3x + 4y$, $-7m - 9x + 3y$ and $5x - 4y$.

26. $a + b + c$, $a + b - c$, $a - b + c$, and $a - b - c$.

27. $a^2 + b^3 + c + d$, $-6a^2 + 3b^3 - d + e$, and $5a^2 - 4b^3 - 3c$.

28. $\frac{1}{2}a + \frac{1}{3}ab + 9b^2$, $\frac{7}{8}a - \frac{2}{11}ab - 15b^2$, $\frac{5}{6}a + \frac{2}{3}ab + 6b^2$, and $-a + ab$.

29. $.25ab + c + .99b$, $3ab + 2c + .01b$, and $.75ab + b$.

30. $7ax + 2bx + 6cx - 3ay + by + 4cy + 2ax + 3bx - cx - 4ay - 2by - 5bx - 5cx + 5ay - 4cy + by$.

31. $3x^3 - 2x^2 + 7x - 1$, $-9x^3 + 11x - 2$, $19x^3 + 7x^2 - 10$, and $15x^3 + 100$.

32. $5a + \frac{1}{2}b - \frac{1}{3}c^2$, $\frac{1}{4}a - 2b + c^2$, $-2a + b - \frac{2}{3}c$.

33. $3x + y - 7z - 6y - 2x - 3z + 15x + 8z - 3y - 9x + 4y + z - y - z - 2x$.

34. $3a^2bx - 5ab^2x + 4abx^2 - 6a^2b^2x$, $3a^2b^2x - 6ab^2x - 2a^2bx - 5abx^2$, and $12ab^2x - a^2bx + 3a^2bx + 3abx^2$.

35. $4\frac{1}{2}x^2y - 3\frac{2}{3}xy^2 + 17x^2y^2$, $13\frac{1}{6}xy^2 - 8\frac{3}{4}x^2y - 21\frac{1}{2}x^2y^2$, $8\frac{1}{4}x^2y + 4\frac{3}{4}x^2y^2 - 8\frac{1}{3}xy^2$, and $2\frac{3}{4}x^2y^2 - 4x^2y - 2\frac{2}{3}xy^2$.

36. $2a^3b^3c^3 - 3a^2b^3c^2 + 12a^2b^3c^3$, $18a^3b^3c^3 - 14a^2b^3c^3 + 5a^3b^3c^2$, $15a^2b^3c^2 - 19a^3b^2c^3 + 13a^2b^3c^3$, and $17a^3b^2c^3 - 12a^2b^3c^2 - 11a^2b^3c^3$.

37. $7x^2 - 3xy - x$, $3x - y^2 + 3x^2$, $4xy - 2x^2 + 5y^2 - y$, $9x - 7xy - y^2$, and $6xy - 11x + y$.

38. $\frac{3}{4}x^2 - 2x + \frac{1}{3}$, $-x^2 + x - 2$, $2x^2 - 5\frac{1}{2}xy - \frac{1}{2}$, $2x + 8\frac{1}{4}xy - 2\frac{3}{4}x^2 + 1\frac{1}{6}$.

SUBTRACTION.

60. Subtraction is the process of finding the simplest expression for the difference between two algebraic quantities.

61. The **minuend** is the quantity from which the subtraction is to be made.

62. The **subtrahend** is the quantity to be subtracted.

63. The **remainder** or **difference** is the quantity obtained by subtracting.

64. The sum of the remainder and subtrahend is equal to the minuend.

65. To find the difference between two quantities.

1. Required to subtract $7 - 4$ from 12.

Explanation.—Subtracting 7 from 12 gives $12 - 7$ or 5. But since not 7 alone, but a quantity 4 less than 7, is to be subtracted from 12, the remainder 5 must be increased by 4.

Process.

$$\begin{aligned} 12 &= 12 \\ 7 - 4 &= 3 \\ \hline 5 + 4 &= 9 \end{aligned}$$

2. Required to subtract $b - c$ from a.

Explanation.—Subtracting b from a gives $a - b$. But since not b alone, but a quantity c less than b, is to be subtracted from a, the remainder $a - b$ must be increased by c. Hence, the

Process.

$$\begin{array}{l} a \\ b - c \\ \hline a - b + c \end{array}$$

66. Rule.—*Conceive the signs of the subtrahend to be changed, and proceed as in addition.*

EXERCISES.

67. *1.* From $25a$ take $12a$.

2. From $10ax$ take $7ax$.

3. From $17x$ take $4x + 5x$.

4. From $5abc + 4abc$ take $3abc$.

5. From $12b$ take $-3b$.

6. From $9ax + 3ax$ take $15ax$.

7. From $2a - 5b$ take $a + 2b$.

8. From $3x - 7y + 4z$ take $2x - 3y + 2z$.

9. From $x - y + z$ take $-x - 2y - 3z$.

10. From $2a - 3b + 2c$ take $-2a + b - 2c$.

11. From $4x^4 - 3x^3 - 2x^2 - 7x + 9$ take $x^4 - 2x^3 - 2x^2 + 7x - 9$.

12. From $8a^3 - 4a^2b - 2bc + 10$ take $3a^3 + a^2b - 5$.

13. From $3b^2c + abx + 2xy^2$ take $b^2c - 3abx + 2xy^2 - 3$.

14. From $8ax - 3by + 4dx$ take $3dx - 5ax + 3by$.

15. From $9bcx + 7aby - 4bx$ take $3bcx + 2aby - 4bx$.

16. From $3bx - 4acy + bxy$ take $2acy - 5bx - bc$.

17. From $9ab - 7de + 8eg$ take $3eg - 7de - 9ab$.

18. From $2x^2 - 3y^2$ take $2x^2 + 3y^2$.

19. From $x^2 + 2xy + y^2$ take $x^2 - 2xy + y^2$.

20. From $x^2 - 2xy - y^2$ take $x^2 - 2xy + y^2$.

21. From $8a + 5b + c$ take $3a + 5b - 3c$.

22. From $5x - 2y + z$ take $4x + y + 3z$.

23. From $x + y$ take the sum of $15x + 2y + 3z$, $-9x - 5y + 2z$, and $-5x + 3y - 5z$.

24. From $ax + bx + cx$ take $-7ax - 3bx + d$.

25. From $9am + yz - xy$ take $3am + 3yz - xy$.

26. From $a + b - c + d$ take the sum of $\frac{1}{2}a + \frac{2}{3}b - \frac{3}{4}c$, $\frac{2}{3}a - \frac{1}{4}c$, and $\frac{1}{3}b + d$.

27. From $x - .99xy - .49x^2$ take $.05x - .001xy + 2.51x^2$.

28. From $\frac{1}{2}x - \frac{2}{3}y + \frac{1}{4}z$ take $\frac{3}{4}x - \frac{2}{3}y - \frac{1}{4}z$.

29. From $2\frac{1}{2}a + 3\frac{1}{3}b - 4\frac{2}{3}c$ take $a - b + \frac{1}{3}c$.

30. From $27 + 6ax - 7by + c$ take $4c + 9 + 5ax - 2by - 5c$.

31. From the sum of $\frac{1}{2}a + b - \frac{3}{4}c$, $\frac{1}{3}a + \frac{1}{2}c$, and $c - a - b$, take $\frac{1}{2}c - \frac{5}{6}a$.

32. From the sum of $.05x - .1y + z$, $.02x + .01y - 1.2z$, and $.09y - .8z - .1x$, take $z - .03x$.

33. From $6ax - 8by + 3c + 4ax + 5by - 6c - 7ax + 9by + 2c$ take $5ax + 3by - 6c - 9ax - 6by + 7c$.

34. From $5c - 3b - 2a - 5b + 4a - 2c + 4b - 5a - 6c$ take $3a - 2b - 9a - 3c + 3b + 2c$.

THE PARENTHESIS.

68. The use of parentheses in algebra is so common that it is important for the pupil early to master the principles pertaining to it.

From the laws governing addition and subtraction we may deduce the following

69. Principles.—I. *A parenthesis preceded by the sign + may be omitted without affecting the value of the expression in which it occurs.*

Thus, $a + (b - c + d) = a + b - c + d$.

II. Conversely: *Any expression or part of an expression may be inclosed in a parenthesis preceded by the sign +.*

Thus, $a + b - c + d = a + (b - c + d)$
$= a + b + (- c + d)$, etc.

III. *A parenthesis preceded by the sign − may be omitted, provided the sign of every quantity within it be changed.*

Thus, $a - (b + c - d) = a - b - c + d$.

IV. Conversely: *Any expression may be inclosed in a parenthesis preceded by the sign −, provided the sign of every quantity inclosed be changed.*

Thus, $a - b + c - d = a - (+ b - c + d) =$
$- (- a + b - c + d) = a - b - (- c + d)$, etc.

V. *Two or more parentheses affecting an expression may be omitted by the application of Principles I. and III.*

Thus, beginning with the inner parenthesis,

$a - [b + \{c - d - (e + f)\}] = a - [b + \{c - d - e - f]\}$
$= a - [b + c - d - e - f] = a - b - c + d + e + f$.

VI. *Any expression may be written with two or more parentheses by the application of Principles II. and IV.*

NOTE.—It is best to introduce the parentheses one at a time, beginning with the inner.

Thus, $a - b + c - d + e = a - b + c - (d - e) =$
$a - b + \{c - (d - e)\} = a - [b - \{c - (d - e)\}]$, etc.

EXERCISES.

70. ***Remove parentheses and reduce to simplest form:***

1. $14 + 4 - (8 - 2)$.
2. $10 - (6 + 3 - 2)$.
3. $13 + 8 - (5 + 12)$.
4. $15 - (12 - 7 + 9)$.
5. $25 - \{18 - (13 - 4)\}$.
6. $100 - [50 - \{30 - (20 - 10)\}]$.
7. $3a - b - (2a - b)$.
8. $32a + 3b - (5a + 17b)$.
9. $a - b + c - (a - b - c)$.
10. $3a + 2b - 3c - (2a - b - c)$.
11. $13a - (5c + 3x - 7a - 5x + 3a)$.
12. $x + (y - z) - (z - 2y + 2x) + (3x - y)$.
13. $-8a + 5b - 3c - (7a - 3b - 2c)$.
14. $3a - 5c + 3d - (7a - 5d + 8c - 2e)$.
15. $37a - 5f - (3a - 2b - 5c) - (-6a - 4b + 3h)$.
16. $a - \{b - c - (d - e)\}$.
17. $2a - (2b - d) - \{a - b - (2c - 2d)\}$.
18. $\{a - (x - y)\} - (2a + x + y)$.
19. $3a - \{b + (2a - b) - (a - b)\}$.
20. $x - [2y + \{3z - 3x - (x + y)\}] + 2x - (y + 3z)$.
21. $ab - \{(ab + ac - a) - (2a - ac) - (2ac - 2a)\}$.
22. $2ab - 3ac - \{(ab - 4ac) + (4a + 2b) + (ac + ab - 4a - 2b)\}$.
23. $ax + b^2 - (d - c) - \{2ax - d + b^2 - (y^2 + ax)\}$.
24. $\{(a - 2b + ab) - (a - b + c)\} - \{a - (a - b + ab)\}$.

MULTIPLICATION.

71. The sum of several equal numbers, as 28, 28, 28, 28, may be found by actually adding them; or it may be found by a process that consists in *remembering* that 4 times 8 units are 32 units, and that 4 times 2 tens are 8 tens, or 80, and then combining 32 and 80 by addition. By the latter process, 28 is said to be *multiplied* by 4, and the result is called the **product.** Hence,

72. Multiplication is a short method of finding the sum of any number of equal quantities.

73. The **multiplicand** is one of the equal quantities whose sum is to be found.

74. The **multiplier** is the quantity that expresses how many of the equal quantities are to be combined.

75. The multiplicand and multiplier are **factors** of the product.

76. To multiply one algebraic quantity by another.

I. THE SIGNS.

1. Required to multiply $8 - 5$ by 4; also $a - b$ by c.

Process.		Explanation.—It is evident that 4 times $8 - 5$ is equal to 4 times 8 minus 4 times 5. In like manner, c times $a - b$ is equal to c times a minus c times b.	Process.
$8 - 5$	$= 3$		$a - b$
4	$= 4$		c
$32 - 20$	$= 12$		$ac - bc$

2. Required to multiply $a - b$ by $c - d$.

Explanation.—Multiplying $a-b$ by c gives $ac - bc$. But the multiplier is not c, but c diminished by d; hence, $ac - bc$ is too large by d times $a - b$.

Diminishing $ac - bc$ by d times $a - b$, and simplifying the result in accordance with the rule for subtraction gives the correct product of $a - b$ by $c - d$.

Process.

$$\begin{array}{l} a - b \\ c - d \\ \hline ac - bc - (ad - bd) = \\ ac - bc - ad + bd. \end{array}$$

77. In the last result above,

1. The term ac shows that *the product of a positive quantity multiplied by a positive quantity is positive.*

2. The term bc shows that *the product of a negative quantity multiplied by a positive quantity is negative.*

3. The term ad shows that *the product of a positive quantity multiplied by a negative quantity is negative.*

4. The term bd shows that *the product of a negative quantity multiplied by a negative quantity is positive.*

78. Hence, in the multiplication of one quantity by another,

1. *Like signs give plus.*
2. *Unlike signs give minus.*

II. THE CO-EFFICIENTS.

3. Required to multiply $7a$ by $5b$.

Explanation.—It is here required to multiply 7 times a by 5 times b; that is, to find the continued product of 7, a, 5, and b. Since the product will be the same in whatever order they are taken, they may be arranged thus, $7 \times 5 \times a \times b = 35 \times ab = 35ab$. Hence,

Process.

$$\begin{array}{l} 7a \\ 5b \\ \hline 35ab \end{array}$$

The co-efficient of the product is equal to the product of the co-efficients of the multiplier and the multiplicand.

III. THE EXPONENTS.

4. Required to multiply a^2b by a^3b^2.

Explanation.—The multiplicand may be written $a \times a \times b$ and the multiplier $a \times a \times a \times b \times b$. Their product, therefore, will be composed of a taken as a factor five times, multiplied by b taken as a factor three times, or $a^5 \times b^3 = a^5b^3$. Hence,

The exponent of any letter in the product is equal to the sum of its exponents in the multiplicand and the multiplier.

79. Letters in the multiplicand or the multiplier, but not in both, must appear in the product with their own exponents.

Thus, $a^2b^3c \times adn^2 = a^3b^3cdn^2$.

80. To multiply a monomial by a monomial.

Rule.—*To the product of the co-efficients of the multiplicand and multiplier annex in alphabetical order all the different literal factors. Give each literal factor an exponent equal to the algebraic sum of its exponents in the multiplicand and multiplier. Prefix to the product the sign + or − according as the signs of the multiplicand and multiplier are like or unlike.*

NOTE.—In finding the continued product of three or more monomials, an odd number of negative monomials will give a negative product, and an even number will give a positive product.

Thus, $-5a \times -3b \times -2c = -30abc$; $5a \times -3b \times -2c = 30abc$.

EXERCISES.

81. ***Find the product of:***

1. $5my$ and $3mx$.
2. $8ay$ and $7a^2y^3$.
3. $4a^2b^2$ and $6ab^3$.
4. $10a^2x^3y$ and $7cxy^2$.
5. $-5a^2b$ and $4ab^2$.
6. $7ab$ and $-5a^2x^3$.
7. $-2ab^2c$ and $5a^2b^3c$.
8. $3a^2x$ and $7a^3x^2y$.
9. $4mn^2$ and $9m^2n$.
10. $-8x^2y$ and $7xy^3$.
11. $-13ac^4$ and $7a^4c$.
12. $25am^3n$ and $5a^3bn^2$.
13. $3ab$, $5a^2c$, $2bc^2$, and $4a^3bc^2x^2$.
14. $-xy$, $2yz$, $-a^2x^2y^3z$, and $12abxy^2$.
15. $-abc$, $-6m^2$, $-3a^2m$, b^2c^3, and $5am^2$.
16. $2a^2x$, $-3xy^2$, $5a^2y$, $-3a^3x^2y^2$, and $-ay^3$.
17. a^2x, xy^3, $a^3x^2y^2$, $2ay$, and $3a^2xy^2$.
18. $2a^2b^2$, $3a^2c^3$, $4b^3c^2$, $5ab^2c^3$, and $-a^2bc$.
19. $13ac^3$ and $4a^3b^6c^2$.
20. $12a^3b^2$ and $5a^2bc^3$.
21. $25a^3x^2$ and $4x$.
22. $20m^2n$ and $5m^3n^2$.
23. $3a^{\frac{3}{4}}bc$ and $2a^{\frac{1}{4}}b^{\frac{1}{4}}c^2$.
24. $-31a^2y$ and $-10ayz^2$.
25. $-15m^2ny$ and $3a^2mn^2$.
26. $3ax$, $-5a^2$, and $2ax^2$.
27. $2m^4n^8x^2$, $3m^7n^9$, and $4m^2n^6x$.
28. $3c^2y$, $4cy^3$, $-acy$, and $2a^2c^3$.
29. $5c^3y^2$, $-4a^2cy$, $-3a^2y^3$, and $2ac^2y$.
30. $3a^2bc^4$, $8ab^2c$, and $2abc^2$.

82. To multiply a polynomial by a polynomial.

1. Required to multiply $x^6 - 5x^4 + 5x^2 - 3$ by $x^4 + 5x^2 + 3$.

Explanation.—For convenience, the multiplicand and multiplier are written so that the exponents in each diminish in value from left to right. The contrary order would do as well. The product of the multiplicand by x^4, the left-hand term of the multiplier, gives the first partial product. The product by $5x^2$ gives the second partial product, which is written so that similar terms shall stand in the same column. The product by 3 gives the last partial product. The sum of these partial products gives the product required.

Process.

$$
\begin{array}{l}
x^6 - 5x^4 + 5x^2 - 3 \\
x^4 + 5x^2 + 3 \\
\hline
x^{10} - 5x^8 + 5x^6 - 3x^4 \\
\quad + 5x^8 - 25x^6 + 25x^4 - 15x^2 \\
\qquad + 3x^6 - 15x^4 + 15x^2 - 9 \\
\hline
x^{10} - 17x^6 + 7x^4 - 9
\end{array}
$$

Rule.—*Multiply the multiplicand by each term of the multiplier and add the partial products.*

EXERCISES.

83. ***Multiply:***

1. $a + b$ by $a + b$.
2. $3x + 2y$ by $2x + 3y$.
3. $3ab + 4b^2$ by $2ab - 3b^2$.
4. $a + b - 2c$ by $2a - b$.
5. $x^2 + y^2 + z$ by $x^2 - y^2$.
6. $a^2 - ab + b^2$ by $a^2 + ab - b^2$.
7. $a^2 - ab + 2b^2$ by $a^2 + ab + 2b^2$.
8. $a^3 - 4a^2 + 11a - 24$ by $a^2 + 4a + 5$.
9. $2x^3 + 4x^2 + 8x + 16$ by $3x - 6$.
10. $27x^3 + 9x^2y + 3xy^2 + y^3$ by $3x - y$.

11. $a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4$ by $a + 2b$.

12. $x^4 + 3x^3 + 5x^2 + 7x + 9$ by $x^2 - 2x + 1$.

13. $a^5 + 3a^4 + 5a^3 + 7a^2 + 9a + 1$ by $a^4 - 3a^3 + 4a^2 - 4a + 4$.

14. $m^4 + 2m^3 + 3m^2 + 4m + 5$ by $m^2 - 2m + 1$.

15. $x^4 + 2x^3 + 3x^2 + 2x + 1$ by $x^2 - 2x + 1$.

16. $2a^4 - 3a^3b - 5a^2b^2$ by $a^3 - 2a^2b + 3ab^2$.

17. $a^5 - a^4b + ab^4 - b^5$ by $a + b$.

18. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

19. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^3 - 3a^2b + 3ab^2 - b^3$.

20. $27a^2 - 13ab + 5b^2$ by $7a^2 + b^2$.

21. $x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4$ by $x - 2y$.

22. $x^5 + 5x^4 + 15x^3 + 15x^2 + 5x + 1$ by $x^4 - 5x^3 + 10x^2 - 5x + 1$.

23. $3x^3 - 15x^2y + 20xy^2 - 12y^3$ by $x^2 + 5xy + 3y^2$.

24. $m^5 + 3m^4 + 4m^3 + 4m^2 + 12m + 21$ by $m^4 - 3m^3 + 5m^2 - 7m + 1$.

25. $42a^4 + 105a^3b + 23a^2b^2 + 5ab^3 + b^4$ by $2a^2 - 5ab + b^2$.

26. $81x^4 + 27x^3y + 9x^2y^2 + 3xy^3 + y^4$ by $3x - y$.

27. $x^5 + 5x^4 + 15x^3 + 30x^2 + 24x + 21$ by $x^4 - 5x^3 + 10x^2 - 5x + 1$.

Find the product of:

28. $x + 4$, $x + 10$, $x - 7$, $x - 9$, and $x + 2$.

29. $x + 8$, $x - 8$, $x + 5$, $x - 5$, $x + 3$, and $x - 3$.

30. $a + b$, $a - b$, $a^2 + ab + b^2$, and $a^2 - ab + b^2$.

31. $x - 4$, $x + 4$, $x + 3$, and $x - 3$.

32. $a^2 + ab + b^2$, $a^3 - a^2b + b^3$, and $a - b$.

33. $1 + x$, $1 + x^4$, and $1 - x + x^2 - x^3$.

84. There are certain cases of multiplication that are important on account of their frequent occurrence in algebraic operations.

85. I. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

Thus, $(x + y)^2 = x^2 + 2xy + y^2$.
$(x + 3)^2 = x^2 + 6x + 9$.

86. ***Write, by inspection, the squares of the following:***

1. $a + b$, $m + n$, $x + 2$, $m + 5$, $n + 7$, $a + 10$, $x + 11$, $y + 8$.

2. $2a + 3b$, $5x + 6y$, $3m + 11n$, $7c + 4d$, $11x + y$, $x + 7y$, $a + 9b$, $x + 20y$.

3. $ax + 5$, $mn + 7$, $pq + 11$, $ac + 13$, $xy + 1$, $7 + 5ab$, $11 + 2mn$.

4. $x^3 + y^2$, $a^2 + m^2$, $a^3 + b^3$, $m^4 + n^4$.

5. $ax + by$, $mn + pq$, $m^3n + xy^2$, $2ab + 3mn$.

87. Numbers may be squared by the foregoing principle.

Thus, $(83)^2 = (80 + 3)^2 = 6400 + 480 + 9 = 6889$.
$(72)^2 = (70 + 2)^2 = 4900 + 280 + 4 = 5184$.
$(104)^2 = (100 + 4)^2 = 10000 + 800 + 16 = 10816$.

88. II. *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

Thus, $(x - y)^2 = x^2 - 2xy + y^2$.
$(x - 4)^2 = x^2 - 8x + 16$.

2. $(x-5)(x-3) = x^2 + (-5-3)x + 5\times 3$
$= x^2 - 8x + 15.$

3. $(x-5)(x+3) = x^2 + (-5+3)x - 5\times 3$
$= x^2 - 2x - 15.$

4. $(x+5)(x-3) = x^2 + (+5-3)x - 5\times 3$
$= x^2 + 2x - 15.$

95. ***Write, by inspection, the products of the following binomials:***

1. $(x+4)(x+3)$, $(x+1)(x+4)$, $(x+2)(x+3)$.
2. $(x+7)(x+2)$, $(x+6)(x+4)$, $(x+9)(x+2)$.
3. $(x-4)(x-1)$, $(x-5)(x-3)$, $(x-7)(x-9)$.
4. $(a-12)(a-10)$, $(m-11)(m-5)$, $(y-8)(y-9)$.
5. $(x-12)(x+5)$, $(y-9)(y+7)$, $(z-11)(z+10)$.
6. $(a-5)(a+4)$, $(b-6)(b+5)$, $(m-12)(m+5)$.
7. $(x+9)(x-7)$, $(x+11)(x-5)$, $(x+12)(x-7)$.
8. $(x+25)(x-4)$, $(x+4)(x-25)$, $(m-30)(m+6)$.

96. Any power of a binomial may be written without actual multiplication. An inspection of the following powers will show the laws governing their formation.

$(x+y)^2 = x^2 + 2xy + y^2.$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3.$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$

$(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5.$

I. *The number of terms in the power is greater by one than the exponent of the power.*

II. *The signs are all + or alternately + and −, according as the sign between the terms of the binomial is + or −.*

III. *The co-efficient of the first term is 1.*

The co-efficient of the second term is equal to the exponent of the power.

The co-efficient of the third term is found by multiplying the co-efficient of the second term by the exponent of x in that term and dividing the product by a number greater by 1 than the exponent of y.

The co-efficient of each succeeding term is found in the same manner from the term preceding.

IV. *The exponent of x in the first term is the same as the exponent of the power, and in each succeeding term it diminishes by 1.*

V. *The exponent of y decreases from the end of the power as x does from the beginning.*

97. If a number is used instead of x or y, the number is treated exactly as the letter would be. Thus,

$$(x+3)^5 = x^5 + 5x^4 \times 3 + 10x^3 \times 3^2 + 10x^2 \times 3^3 + 5x \times 3^4 + 3^5$$
$$= x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243.$$

98. ***Write, by inspection, the following powers:***

1. $(a+b)^4$.	*6.* $(d+e)^4$.	*11.* $(x-y)^7$.
2. $(a-b)^4$.	*7.* $(d-e)^6$.	*12.* $(y-z)^3$.
3. $(a+c)^3$.	*8.* $(a-2c)^3$.	*13.* $(m+n)^4$.
4. $(b-d)^2$.	*9.* $(y+z)^9$.	*14.* $(m-n)^3$.
5. $(a-c)^3$.	*10.* $(x+y)^3$.	*15.* $(p+q)^5$.

DIVISION.

99. Division in algebra is the process of finding one of two algebraic factors when their product and the other factor are given.

100. The **dividend** is the given product.

101. The **divisor** is the given factor.

102. The **quotient** is the required factor.

103. To divide one algebraic quantity by another.

I. THE SIGNS.

Since $(+a) \times (+b) = +ab$, it follows that

$$(+ab) \div (+a) = +b.$$

Since $(-a) \times (-b) = +ab$, it follows that

$$(+ab) \div (-a) = -b.$$

Hence,

1. **Like signs** *in the dividend and the divisor give* $+$ *in the quotient.*

2. **Unlike signs** *in the dividend and divisor give* $-$ *in the quotient.*

II. THE CO-EFFICIENTS AND EXPONENTS.

Since $7a^5b^3 \times 5a^2b = \overline{7 \times 5}a^{5+2}b^{3+1}$, it follows that

$$35a^7b^4 \div 5a^2b = \overline{35 \div 5}\, a^{7-2}\, b^{4-1} = 7a^5b^3.$$

Hence,

1. *The co-efficient of the dividend divided by the co-efficient of the divisor gives the co-efficient of the quotient.*

2. *The exponent of a letter in the dividend diminished by the exponent of that letter in the divisor gives its exponent in the quotient.*

104. The literal factors of the dividend that are not in the divisor must appear in the quotient.

Thus, $a^2b^3c \div ab = ab^2c$.

105. Since $a^2 \div a^2 = a^{2-2} = a^0$; and since any quantity whatever divided by itself gives 1, it follows that $a^0 = 1$.

That is, *any quantity with zero for an exponent is equal to 1.*

106. To divide when the divisor is a monomial.

1. Required to divide $6a^2b^5c - 9a^3b^5 + 15a^5b^3n$ by $-3a^2b^3$.

Process.

$$-3a^2b^3 \;)\; 6a^2b^5c - 9a^3b^5 + 15a^5b^3n \;(\; -2b^2c + 3ab^2 - 5a^3n$$

Explanation.—In dividing $6a^2b^5c$ by $-3a^2b^3$, the sign in the quotient will be $-$, since the signs in the dividend and divisor are unlike; the co-efficient in the quotient will be $6 \div 3$, or 2; and the literal part will be $a^{2-2}b^{5-3}c = b^2c$. In dividing $-9a^3b^5$ by $-3a^2b^3$, the sign in the quotient will be $+$, since the signs are like; the co-efficient will be $9 \div 3$, or 3; and the literal part will be $a^{3-2}b^{5-3} = ab^2$. In the same manner the other term is found.

Rule.—*Write the divisor at the left of the dividend and divide each term in succession of the dividend by the divisor.*

EXERCISES.

107. *1.* Divide acd by c.

2. Divide $16a^5b^4x^3$ by $8ab^2x^2$.

3. Divide $7abm$ by am.

4. Divide $6a^4b^2cd$ by $3ab$.

5. Divide $-14a^2x^3y^4$ by $-7ax^3y^3$.

6. Divide $-36a^2b^2c$ by $18ab$.

7. Divide $15x^2y^3 + 30x^5y^7$ by $5x^2$.

8. Divide $8a^3bc + 16a^5bc - 4a^2c^2$ by $4a^2c$.

9. Divide $20a^4bc + 15abd^3 - 10a^2be$ by $5ab$.

10. Divide $9a^5bc - 3a^2b + 18a^3bc$ by $3ab$.

11. Divide $-9a^2bc - 12ab^2c + 15abc^2$ by $-3abc$.

12. Divide $36m^3 - 42m^2n$ by $6m^2$.

13. Divide $45a^2b^2c - 60abc^2$ by $15abc$.

14. Divide $5a^2b + 20a^3b^2n - 35abn^2$ by $5ab$.

15. Divide $9x^2yz - 36xy^2z^3 + 45axyz^5$ by $9xyz$.

16. Divide $a^3b^2c + a^2bc - a^3b^2c^2 + a^4b^4c^3$ by a^2bc.

17. Divide $15ax^3 - 25bx^2y + 35cxy^2$ by $-5x$.

18. Divide $m^2nx^4y^2 - mn^2x^2y^4$ by $mnxy$.

19. Divide $6a^3b^3 - 10a^2c + 7a^4bx$ by $2a^2$.

20. Divide $5x^3y^3 - 40a^2x^2y^2 + 25a^4xy$ by $-5xy$.

21. Divide $9a^4b^2c^3 - 6a^5b^3c^4 - 3a^6b^4c^5$ by $3abc$.

22. Divide $24x^2y^2 - 8x^4y^5 - 24xy^2 - 40x^2y^3$ by $8x$.

23. Divide $21a^3x^3 - 7a^2x^2 + 14ax - 63$ by $7ax$.

24. Divide $42a^3 - 11a^2 + 28a + 35$ by $7a$.

108. To divide when the divisor is a polynomial.

I. Required to divide $x^2 + 2xy + y^2$ by $x + y$. Also, $a^2 - 2ab + b^2$ by $a - b$.

$$\begin{array}{l|l} x^2 + 2xy + y^2 & x + y \\ \underline{x^2 + xy} & x + y \\ \ xy + y^2 & \\ \ \underline{xy + y^2} & \end{array} \qquad \begin{array}{l|l} a^2 - 2ab + b^2 & a - b \\ \underline{a^2 - ab} & a - b \\ \ - ab + b^2 & \\ \ \underline{- ab + b^2} & \end{array}$$

II. Required to divide $x^2 - y^2$ by $x + y$. Also, $a^4 - x^4$ by $a^2 - x^2$.

$$\begin{array}{l|l} x^2 - y^2 & x + y \\ \underline{x^2 + xy} & x - y \\ \ - xy - y^2 & \\ \ \underline{- xy - y^2} & \end{array} \qquad \begin{array}{l|l} a^4 - x^4 & a^2 - x^2 \\ \underline{a^4 - a^2x^2} & a^2 + x^2 \\ \ a^2x^2 - x^4 & \\ \ \underline{a^2x^2 - x^4} & \end{array}$$

III. Required to divide $a^4 + a^2x^2 + x^4$ by $a^2 + ax + x^2$.

$$\begin{array}{l|l} a^4 + a^2x^2 + x^4 & a^2 + ax + x^2 \\ \underline{a^4 + a^3x + a^2x^2} & a^2 - ax + x^2 \\ \ - a^3x + x^4 & \\ \ \underline{- a^3x - a^2x^2 - ax^3} & \\ \ a^2x^2 + ax^3 + x^4 & \\ \ \underline{a^2x^2 + ax^3 + x^4} & \end{array}$$

IV. Required to divide $x^5 - 9 + 7x^2 - 17x^3$ by $3 + x^2 + 5x$.

$$\begin{array}{l|l} x^5 - 17x^3 + 7x^2 - 9 & x^2 + 5x + 3 \\ \underline{x^5 + 5x^4 + 3x^3} & x^3 - 5x^2 + 5x - 3 \\ \ - 5x^4 - 20x^3 + 7x^2 & \\ \ \underline{- 5x^4 - 25x^3 - 15x^2} & \\ \ 5x^3 + 22x^2 - 9 & \\ \ \underline{5x^3 + 25x^2 + 15x} & \\ \ - 3x^2 - 15x - 9 & \\ \ \underline{- 3x^2 - 15x - 9} & \end{array}$$

Rule.—I. *Arrange the terms of the dividend and divisor in the order of the ascending or descending powers of the same letter.*

II. *Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.*

III. *Multiply the entire divisor by the first term of the quotient; write the product under the dividend, and subtract it from the dividend for the first remainder.*

IV. *Divide the first term of the remainder by the first term of the divisor for the second term of the quotient.*

V. *Multiply the divisor by this term, write the product under the first remainder, and subtract as before.*

VI. *So continue until the last remainder is 0, or until the first term of the remainder will not contain the first term of the divisor without a change in the sign of the exponent of the quotient.*

NOTE.—When there is a final remainder, it is to be written over the divisor, and annexed to the quotient in order to complete it.

EXERCISES.

109. ***Divide:***

1. $x^2 + 2xy + y^2$ by $x + y$.
2. $x^2 - 2xy + y^2$ by $x - y$.
3. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x - y$.
4. $x^3 + x^2 + 4x - 20$ by $x - 2$.
5. $x^4 + 2x^3 - 3x^2 - 4x + 4$ by $x - 1$.
6. $2x^3 - x^2 + 3x - 9$ by $2x - 3$.
7. $x^4 - 9x^2 - 6xy - y^2$ by $x^2 + 3x + y$.
8. $8a^2 - 26ab + 15b^2$ by $4a - 3b$.
9. $x^3 + 6x^2 + 9x + 4$ by $x + 4$.

10. $x^3 - 3x + 2$ by $x^2 - 2x + 1$.

11. $x^3 - 5x^2 - x + 14$ by $x^2 - 3x - 7$.

12. $a^4 - a^2b^2 - 12b^4$ by $a^2 + 3b^2$.

13. $x^3 - 86x - 140$ by $x - 10$.

14. $x^3 - 5x^2 - 46x - 40$ by $x + 4$.

15. $7x^3 - 24x^2 + 58x - 21$ by $7x - 3$.

16. $x^4 - 2x^3y + 2x^2y^2 - xy^3$ by $x - y$.

17. $x^4 - 5x^3 + 11x^2 - 12x + 6$ by $x^2 - 3x + 3$.

18. $x^4 + 64$ by $x^2 + 4x + 8$.

19. $27x^3 + 8y^3$ by $3x + 2y$.

20. $x^4 - 16$ by $x - 2$.

21. $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.

22. $7a^6 - 48a^4b^2 + b^6$ by $a^2 - 3ab + b^2$.

23. $a^4 - a^2b^2 + 4ab^3 - 4b^4$ by $a^2 - ab + 2b^2$.

24. $6x^4 - 96$ by $3x - 6$.

25. $x^6 + 10x - 33$ by $x^2 - 2x + 3$.

26. $x^4 - 81y^4$ by $x - 3y$.

27. $x^4 - 6x^3y + 9x^2y^2 - 4y^4$ by $x^2 - 3xy + 2y^2$.

28. $6x^3 + 4y^3 + 6xy^2 - 16x^2y$ by $3x^2 - y^2 - 2xy$.

29. $x^6 - 6x + 5$ by $x^3 - 2x + 1$.

30. $1 - x - 3x^2 - x^5$ by $1 + 2x + x^2$.

31. $a^2b^2 - a^2 - b^2 + 1$ by $ab + a + b + 1$.

32. $a^2b^2 + 2ab^2c - a^2c^2 + b^2c^2$ by $ab + ac + bc$.

33. $12a^4 - 26a^3b - 8a^2b^2 + 10ab^3 - 8b^4$ by $3a^2 - 2ab + b^2$.

34. $3a^2 + 8ab + 4b^2 + 10ac + 8bc + 3c^2$ by $a + 2b + 3c$.

35. $x^3 + y^3 + z^3 - 3xyz$ by $x^2 + y^2 + z^2 - xy - xz - yz$.

36. $a^6 - b^6$ by $a^3 - 2a^2b + 2ab^2 - b^3$.

37. $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$ by $1 - 3x + 3x^2 - x^3$.

38. $1 - 81y^4$ by $1 - 3y$.

39. $a^5 + 31a - 56$ by $a^2 + 4a + 7$.

40. $x^6 - 6x + 5$ by $x^2 - 2x + 1$.

41. $x^6 + 2x^5 - 4x^4 - 2x^3 + 12x^2 - 2x - 1$ by $x^3 + 2x - 1$.

42. $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$ by $x^4 - 2x^3 + 3x^2 - 2x + 1$.

43. $x^6 - 13x^4 - 12x + 8$ by $x^3 + 3x - 2$.

44. $2x^6 - 9x^4 - 8x^3 - 1$ by $x^3 + 3x^2 + 3x + 1$.

45. $3x^5 - 46x^3 + 43x^2 - 36$ by $x^3 + 5x + 3$.

46. $a^{10} + 16a^4 - 5a^3 - 15a - 5$ by $a^4 + 2a^3 - 3a - 1$.

47. $x^{10} - 29x^4 + 62x^3 + 3x^2 + 16x - 352$ by $x^4 + 2x^3 - 5x - 11$.

48. $x^7 + 389x^2y^5 - 149xy^6 + 115y^7$ by $x^3 + 3x^2y - xy^2 + y^3$.

49. $32x^3 + 4x^3 - 2x^2 - 1$ by $4x^3 + 4x^2 + 2x + 1$.

50. $m^{10} - 6m^3 + 5m - 2$ by $m^4 + 2m^3 - 3m - 2$.

51. $x^8 - 51x^5y^3 - 56x^2y^6 + 186xy^7 - 95y^8$ by $x^3 - 3x^2y - 4xy^2 + 5y^3$.

52. $a^8 - 7a^7 + 49a^5 + 24a^3 + 52a^2 - 19a + 20$ by $a^3 - 5a^2 + 3a - 4$.

53. $m^6 - 57m^4n^2 + 58m^2n^4 - 9n^6$ by $m^3 + 7m^2n - 4mn^2 - 3n^3$.

There are certain cases of division which, on account of their frequent occurrence, are important.

110. PRINCIPLE I.—*The sum of equal odd powers of two quantities is exactly divisible by the sum of the quantities.*

Thus, $(x^3 + y^3) \div (x + y) = x^2 - xy + y^2$.

$(x^5 + y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

$(a^3 + 8) \div (a + 2) = a^2 - 2a + 4$.

$(a^5 + 243) \div (a + 3) = a^4 - 3a^3 + 9a^2 - 27a + 81$.

111. ***Determine an exact divisor and the quotient for each of the following, by inspection:***

1. $x^3 + a^3$, $m^3 + n^3$, $a^3 + 1$, $m^3 + 8$, $x^3 + 27$, $x^3 + 125$.
2. $8 + n^3$, $1 + x^3$, $a^3 + 64$, $y^3 + 216$, $m^3 + 1000$.
3. $m^5 + n^5$, $x^5 + a^5$, $x^5 + 1$, $1 + m^5$, $32 + n^5$, $x^5 + 243$.
4. $a^3x^3 + 1$, $m^3 + x^3y^3$, $8a^3 + 1$, $27x^3 + a^3x^3$, $a^3y^3 + 64$.
5. $a^5 + m^5x^5$, $a^5x^5 + 1$, $a^3y^3 + 1000$, $8x^3y^3 + 729$, $a^3b^3 + x^3y^3$.

112. PRINCIPLE II.—*The difference of equal odd powers of two quantities is exactly divisible by the difference of the quantities.*

Thus, $(x^3 - y^3) \div (x - y) = x^2 + xy + y^2$.

$(x^5 - y^5) \div (x - y) = x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

$(a^3 - 8) \div (a - 2) = a^2 + 2a + 4$.

$(x^5 - 243) \div (x - 3) = x^4 + 3x^3 + 9x^2 + 27x + 81$.

113. ***Find an exact divisor for each of the following binomials, and write the quotient, by inspection:***

1. $x^3 - 27$, $a^3 - b^3$, $m^3 - 1$, $x^3 - 125$, $216 - x^3$.

2. $8 - x^3$, $1 - 8y^3$, $a^3 - 1000$, $a^3b^3 - 1$, $x^3y^3 - 8$.

3. $m^5 - n^5$, $x^5 - 32$, $x^5 - 1$, $a^5 - 243$, $1 - y^7$, $y^9 - 1$.

4. $m^3n^3 - 8x^3$, $p^5q^5 - 1$, $125x^3 - a^3y^3$, $64m^3 - 8n^3$, $32x^5 - y^5$.

5. $x^5 - 243y^5$, $x^7 - y^7$, $x^9 - 1$, $a^7b^7 - 1$, $a^3b^3 - 64x^3$, $a^5 - x^5y^5$.

114. PRINCIPLE III.—*The difference of equal even powers of two quantities is exactly divisible either by the sum or by the difference of the quantities.*

Thus, $(x^4 - y^4) \div (x + y) = x^3 - x^2y + xy^2 - y^3$.

$(x^4 - y^4) \div (x - y) = x^3 + x^2y + xy^2 + y^3$.

$(x^5 - y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

$(x^5 - y^5) \div (x - y) = x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

115. ***Divide the following in accordance with Principle III., by inspection.***

1. $x^2 - y^2$, $a^4 - b^4$, $m^6 - n^6$, $x^8 - y^8$, $x^2 - 64$, $x^4 - 1296$.

2. $x^4 - 81$, $x^4 - 625$, $x^6 - 64$, $m^4 - 16$, $m^6 - 64$, $x^8 - 1$.

3. $x^6 - 1$, $x^4 - 256$, $1 - x^6$, $16 - 81x^4$, $a^2b^2 - 64$.

4. $a^4x^4 - 1$, $a^4x^4 - m^4n^4$, $16a^4 - 81$, $625x^4 - 81y^4$, $x^2y^2 - 16z^2$.

FACTORING.

116. A **factor** of a quantity is any quantity that will exactly divide it.

117. **Factoring** is the process of finding such divisors of a quantity as will produce it when multiplied together.

PRINCIPLES OF FACTORING.

118. I. *Any polynomial containing a common factor in each term may be factored by dividing the polynomial by the common factor.*

Thus, $5a^3b^2 - 15a^2b^3 + 10a^3b^2 = 5a^2b^2(a - 3b + 2)$.

119. ***Find the factors of the following:***

1. $3a^2b + 9a^3b^2$.
2. $6x^2y^3 - 9x^3y^2$.
3. $9m^2n^4 - 15m^2n^2x$.
4. $15a^3b^3x^2 + 25a^2b^3x^3$.
5. $27x^2y^3z - 36xy^2z^2$.
6. $45am^2n + 60m^2nx^2$.
7. $a^2b^2c - ab^2c^2 + a^3bc^3$.
8. $5am^2 + 10bmn + 15cmn^2$.
9. $9x^2y^3z - 12x^3y^2z^2 + 15xyz^2$.
10. $18m^2n^5x^3 - 24m^3n^3x^5 + 30m^4n^5x^4$.
11. $39a^4b^6c^3 - 52a^4b^3c^7 + 91a^3b^6c^4$.
12. $90x^4y^3z^4 + 180x^5y^3z^3v^2 - 225x^3y^2z^4$.
13. $39x^2y^3z^3 - 65x^3yz^2 + 91x^2y^3z^3 - 117x^4y^4z^4$.
14. $36a^2b - 60ab^2 + 84abc^2 - 120a^3b^2x - 72a^2b^3y + 144ab^2c^3$.

120. II. *Any binomial expressing the difference between squares may be separated into two binomial factors, one of which is the sum of their square roots, and the other the difference.*

Thus, $a^2 - b^2 = (a + b)(a - b)$;

$a^4 - 4 = (a^2 + 2)(a^2 - 2)$.

$x^4 - 16 = (x^2 + 4)(x^2 - 4)$

$= (x^2 + 4)(x + 2)(x - 2)$.

121. *Factor the following:*

1. $x^2 - y^2$, $x^4 - y^4$, $a^2 - 9$, $a^4 - 81$, $225 - 64x^2y^4$.
2. $m^2 - n^2$, $x^2 - a^2$, $a^6 - b^6$, $p^8 - q^8$, $16a^4 - 81b^4$.
3. $a^4b^2 - a^2b^4$, $a^2b^4 - 16$, $x^4y^2 - 25$, $x^6y^4 - 81a^4b^6$.
4. $a^6 - 625b^4$, $256m^4 - 1$, $1 - 81m^4n^2$, $49m^4 - 1$.
5. $a^6 - 729$, $x^8 - y^8$, $m^2n^6 - x^6y^4$, $m^4n^4 - 256$.
6. $9a^2 - 25b^2$, $16 - 25x^2$, $81a^4b^2 - 64a^2x^4$.
7. $4a^2 - 9x^2$, $16b^2 - 25c^2$, $49a^2 - 64x^2$, $81y^2 - 100z^2$.
8. $81a^2 - 25$, $49 - b^2c^2$, $16x^2 - 4y^4$, $9a^2 - 16c^2$.

122. III. *Any binomial expressing the sum of equal odd powers may be separated into two factors, one of which is the sum of the quantities.*

The other factor may be obtained by division, and, after some practice, by inspection.

Thus, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.

$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$.

$1 + 32x^5 = (1 + 2x)(1 - 2x + 4x^2 - 8x^3 + 16x^4)$.

123. ***Factor the following :***

1. $x^3 + y^3$, $a^5 + b^5$, $m^3 + 8$, $p^3 + q^3$, $a^3 + 216$.
2. $x^3 + 1$, $a^3 + 8$, $m^3 + 27$, $125 + n^3$, $64 + x^3$.
3. $x^5 + 1$, $1 + a^5$, $m^5 + 32$, $243 + b^5$.
4. $a^3b^3 + x^3y^3$, $8m^3 + 27n^3$, $125x^3y^3 + 64b^3$, $32m^5 + 1$.

124. IV. *Any binomial expressing the difference between equal odd powers may be separated into two factors, one of which is the difference between the quantities.*

Thus, $x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$.

125. ***Factor the following :***

1. $x^3 - y^3$, $x^5 - a^5$, $x^7 - b^7$, $a^9 - b^9$, $a^3 - 8$.
2. $x^3 - 1$, $x^3 - 8$, $x^3 - 27$, $x^3 - 125$, $a^3 - 64$.
3. $a^5 - 32$, $x^7 - 128$, $1 - x^3$, $125 - a^3$, $216 - y^3$.
4. $8x^3 - 27y^3$, $64m^3 - 125n^3$, $343 - m^3$, $512 - m^3$.
5. $216m^3 - 512n^3$, $32x^5 - 1$, $a^3x^3 - 125$, $x^3y^3 - 8z^3$.
6. $a^3x^3 - b^3$, $m^3 - 8a^3x^3$, $32m^5n^5 - 1$, $a^5b^5 - m^5n^5$.

126. V. *Any trinomial of which two terms are positive squares, and the third equal to twice the product of the square roots of the other two may be separated into two equal binomial factors.*

Thus,
$$a^2 + 2ab + b^2 = (a + b)(a + b).$$
$$a^2 - 2ab + b^2 = (a - b)(a - b).$$
$$x^2 - 20x + 100 = (x - 10)(x - 10).$$
$$16x^2 + 56x + 49 = (4x + 7)(4x + 7).$$
$$9x^4 - 30x^2y + 25y^2 = (3x^2 - 5y)(3x^2 - 5y).$$

127. ***Factor the following:***

1. $x^2 + 2xy + y^2$, $m^2 - 2mn + n^2$, $a^4 - 2a^2b^2 + b^4$.
2. $x^4 + 2x^2y + y^2$, $a^8 - 2a^4b^2 + b^4$, $a^{16} - 2a^8b + b^2$.
3. $a^2 - 18a + 81$, $4a^2 + 4ab + b^2$, $4x^2 - 8xy + 4y^2$.
4. $a^2 - 18ab + 81b^2$, $121x^2 + 220xy + 100y^2$.
5. $225m^4 - 30m^2n^2 + n^4$, $36x^2 - 84xy + 49y^2$.
6. $x^4 - 2x^2y + y^2$, $4x^8 + 12x^4y^2 + 9y^4$.
7. $25m^4n^2 - 30m^2nx^2 + 9x^4$, $25a^2 + 60ab + 36b^2$.
8. $49m^6 - 140m^3n^2 + 100n^4$, $225x^2 + 360xy^2 + 144y^4$.
9. $x^4y^2 - 4x^2yz^3 + 4z^6$, $4a^4x^6 + 20a^2x^3y^4 + 25y^8$.
10. $81b^4c^2 - 126b^2cd^3 + 49d^6$, $121a^2y^4 - 220ay^2z + 100z^2$.
11. $400a^2 - 1200ab^2 + 900b^4$, $1600m^2 - 4000mn^3 + 2500n^6$.

128. VI. *Any trinomial whose terms are positive, the extremes being squares and the middle term the product of the square roots of the extremes, may be separated into two trinomial factors.*

Thus, $a^4 + 4a^2 + 16 = (a^2 + 2a + 4)(a^2 - 2a + 4)$.

$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2)$.

$m^8 + 25m^4 + 625 = (m^4 + 5m^2 + 25)(m^4 - 5m^2 + 25)$.

129. ***Factor the following:***

1. $a^4 + a^2y^2 + y^4$, $m^4 + m^2n^2 + n^4$, $p^4 + p^2q^2 + q^4$.
2. $x^4 + 9x^2 + 81$, $16a^4 + 4a^2x^2 + x^4$, $x^8 + x^4y^4 + y^8$.
3. $x^8 + x^4y^2 + y^4$, $m^4 + m^2n^4 + n^8$, $81a^4 + 36a^2 + 16$.
4. $p^8 + p^4 + 1$, $m^4 + m^2 + 1$, $x^4 + 16x^2 + 256$.

130. VII. *Any trinomial may be separated into two binomial factors when one of its terms is the product of the square root of one of the others by the algebraic sum of two factors of the third.*

The binomials will be the algebraic sum of this square root and each of the factors respectively.

Thus, $a^2 + 7a + 10 = (a + 5)(a + 2)$. Here $7a$ is the product obtained by multiplying the square root of a^2 by the sum of two factors of $+ 10$.

Again, $a^2 - 7a + 10 = (a - 5)(a - 2)$. In this case the two factors of $+ 10$, whose sum multiplied by a equals $- 7a$, are $- 5$ and $- 2$.

Again, $x^2 - x - 12 = (x - 4)(x + 3)$. Here the two factors of $- 12$ are $- 4$ and $+ 3$, whose algebraic sum is $- 1$.

131. ***Factor the following:***

1. $x^2 + 3x + 2$, $x^2 + 10x + 21$, $a^2 + 8a + 12$.
2. $x^2 + 15x + 56$, $x^2 + 21x + 20$, $m^2 + 13m + 42$.
3. $x^2 + 14x + 40$, $x^2 + 11x + 28$, $m^2 + 17m + 60$.
4. $a^2 - 9a + 20$, $a^2 - 11a + 24$, $m^2 - 10m + 21$.
5. $x^2 - 9x + 8$, $x^2 - 15x + 36$, $x^2 - 20x + 96$.
6. $x^2 - 19x + 84$, $x^2 - 15x + 44$, $x^2 + 11x - 102$.
7. $y^2 - 2y - 15$, $y^2 - 4y - 77$, $x^2 - 7x - 60$.
8. $x^2 + 11x - 42$, $x^2 + 11x - 60$, $x^2 + 11x - 210$.
9. $x^4 - 9x^2 - 22$, $a^4 + 5a^2 - 24$, $x^4 - 8x^2 - 65$.
10. $a^2 + 27a + 50$, $y^2 - 5y - 50$, $x^2 + 43x - 90$.
11. $a^2 + 8a + 15$, $c^4 + 20c^2 - 69$, $b^4 - 2b^2 - 120$.

132. VIII. *A trinomial may often be factored by arranging the factors of its extreme terms.*

To show this, let us take $10x^2 + 19x + 6$.

We naturally suppose,

1. That it is the product of two binomials.

2. That its first term, $10x^2$, is the product of the first terms of these binomials.

3. That its last term, 6, is the product of the second terms of the binomials.

4. That its middle term is the algebraic sum of the products of these extremes taken diagonally.

5. That the signs of the binomials determine those of the trinomial.

Taking $5x$ and $2x$ as the most probable factors of $10x^2$, and 2 and 3 as those of 6, we consider how we shall arrange them for multiplication, so that the algebraic sum of the cross products shall be $19x$. A brief trial leads to the following arrangement and signs:

$$5x + 2,$$
$$2x + 3.$$

The factors are, therefore, $(5x + 2)(2x + 3)$.

Had every arrangement of these factors failed, $10x^2$ and 6 might have been factored differently. A little practice will enable the pupil to factor such trinomials by inspection alone.

133. *Factor the following:*

1. $5x^2 + 11x + 2$, $5x^2 + 7x + 2$, $2y^2 + 11y + 15$.
2. $3a^2 + 22a + 7$, $5x^2 + 36x + 7$, $11m^2 + 34m + 3$.

3. $2x^2 - 17x + 21$, $2n^2 - 17n + 35$, $2n^2 - n - 21$.

4. $15m^2 - 58m + 11$, $39x^2 - 34x + 7$, $3y^2 - y - 14$.

5. $15b^2 + 44b - 3$, $7y^2 - 34y + 39$, $15x^2 - 7x - 30$.

6. $2x^2 + x - 66$, $3y^2 + 2y - 65$, $21a^2 + 11a - 2$.

7. $91m^2 - 10m - 21$, $15a^2 - a - 6$, $5x^2 - 4x - 1$.

8. $2x^4 - x^2 - 10$, $15n^4 - n^2 - 28$, $22x^6 - 41x^3 - 35$.

9. $4m^4 - 25m^2 + 36$, $8x^6 - 19x^3 - 27$, $6x^4 - 5x - 6$.

134. IX. *A polynomial may sometimes be factored by grouping its terms.*

Thus, a brief inspection of the expression

$$mx + am + 2mc + nx + an + 2nc,$$

will show the propriety of the following groups:

$$(mx + nx) + (am + an) + (2mc + 2nc),$$

where $m + n$ occurs in each group. The transition to the expression

$$x(m + n) + a(m + n) + 2c(m + n),$$

and finally to $(x + a + 2c)(m + n)$ is evident.

In like manner, $2ax + 3bx - cx - 4ay - 6by + 2cy$ becomes successively,

$$(2ax - 4ay) + (3bx - 6by) - (cx - 2cy),$$

$$2a(x - 2y) + 3b(x - 2y) - c(x - 2y),$$

and

$$(2a + 3b - c)(x - 2y).$$

135. ***Factor the following:***

1. $mx + 2x + 2y + my$, $ax + bx + a + b$.
2. $4x + 2ax + 3a + 6$, $ac + bd + ad + bc$.
3. $a - 3b + ab - 3$, $3xy - y - 6x + 2$.
4. $2x^2 - 3my - 2mx + 3xy$, $2x^2 - 3m - 6x + mx$.
5. $mq^2 + x^2 + pq - qx - mqx - px$.
6. $ap - cq + cs + aq - cp - as$.
7. $ar - 2bc + 2br - ac + 4bn + 2an$.
8. $ax - cx + 2by + ay - cy + 2bx$.
9. $2cd + 6bc - 12abc - 4acd + ad + 3ab$.
10. $bd - bm + 2am - cm + cd - 2ad$.
11. $3an - 4br - 6bn + 2ar - 8bc + 4ac$.
12. $an + 2brs - acd - bcds + 2ar + bns$.
13. $2ax - cz - ay + 2cx - cy - az$.
14. $6sy + 9ny - 2px + 6nx + 4sx - 3py$.
15. $2y - 6ay - 2x - 4by + 6ax + 4bx$.
16. $21x - 5n + 3nx - 35 - 14y - 2ny$.

136. X. *The continued product of several binomial factors whose first terms are alike, may be factored.*

Thus, $(x + a)(x + b)(x + c)$
$= x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$.

$(x + a)(x - b)(x - c)$
$= x^3 + (a - b - c)x^2 + (-ab - ac + bc)x + abc$.

$(x + 5)(x - 3)(x + 1)$
$= x^3 + (5 - 3 + 1)x^2 + (-15 + 5 - 3)x - 15$
$= x^3 + 3x^2 - 13x - 15$.

1. The first term is the continued product of the first terms of the factors.

2. The exponent of x diminishes by 1 in each succeeding term, and disappears in the last term.

3. The co-efficient of the second term is the algebraic sum of the second terms of the factors.

4. The co-efficient of the third term is the algebraic sum of all the products of the second terms of the factors taken two at a time.

5. The last term is the algebraic product of the second terms of the factors.

Assuming that $x^3 + 4x^2 - 17x - 60$ is the product of three binomial factors of the form $x \pm a$, let it be required to find those factors.

1. The first term of each is x.

2. The continued product of the second terms must be -60, and their algebraic sum $+4$.

3. The algebraic sum of all the products of the second terms taken two at a time is -17.

4. One, but not three, of the factors of -60 must be negative, since their algebraic sum is $+4$.

Taking 3, 4, and 5 as the factors of 60 most likely to be the second terms of the binomials, we determine which shall be minus that their sum may be $+4$. By making them $+3$, -4, and $+5$, the requirement is met.

It remains now to see whether the sum of their products taken two at a time is -17.

$$\begin{aligned}(+3)\times(-4) &= -12\\ (+3)\times(+5) &= +15\\ (-4)\times(+5) &= -20\\ \hline &+15-32=-17.\end{aligned}$$

Hence, $x^3+4x^2-17x-60=(x+3)(x-4)(x+5)$.

137. ***Factor the following:***

1.	$x^3+6x^2+11x+6$.	*9.*	x^3+8x^2-x-8.
2.	$x^3+5x^2-22x+16$.	*10.*	$x^3-28x-48$.
3.	$x^3+5x^2-2x-24$.	*11.*	$x^3-27x+54$.
4.	$x^3+12x^2+5x-150$.	*12.*	$x^3-75x+250$.
5.	$x^3-x^2-37x+35$.	*13.*	$x^3-39x-70$.
6.	$x^3+4x^2-11x-30$.	*14.*	$x^3-17x^2-185x-375$.
7.	$x^3-3x^2-25x-21$.	*15.*	$x^3+19x^2+103x+165$.
8.	$x^3-10x^2+31x-30$.	*16.*	$x^3-2x^2-89x+90$.

138. XI. *A polynomial consisting of the square of a binomial and some multiple of that binomial may easily be factored.*

The method of factoring them will be seen from their formation.

$$(2x+5)^2+4y(2x+5)=4x^2+20x+25+8xy+20y.$$
$$(3x-2)^2-5m(3x-2)=9x^2-12x+4-15mx+10m.$$
$$(a^2-2b)^2-cx(a^2-2b)=a^4-4a^2b+4b^2-a^2cx+2bcx.$$

Reversing this process with the expression,

$$\begin{aligned}m^4-6m^2n+9n^2-6m^2p+18np&\\ &=(m^2-3n)(m^2-3n)-6p(m^2-3n)\\ &=(m^2-3n-6p)(m^2-3n).\end{aligned}$$

139. ***Factor the following:***

1. $x^2 + 2ax + a^2 + 6x + 6a.$
2. $x^2 - 4ax + 4a^2 + 9x - 18a.$
3. $m^2 - 10m + 25 - mn + 5n.$
4. $x^2 - 14x - 5ax + 35a + 49.$
5. $a^2 + 9x^2 - 6ax + 2a - 6x.$
6. $by^2 + y^3 + byz + z^2 + 2yz.$
7. $b^3 - 4ab - 3b^2c + 4ac + 3bc^2 - c^3.$
8. $6mr + 25n^2 - 30nr - 10mn + m^2.$
9. $16y^2 - 15xz + 9x^2 - 24xy + 20yz.$
10. $21xy + 49x^2 - 6y + 4 - 28x.$
11. $m^4 - 6m^2n^2 + 9n^4 - 6m^2p + 18n^2p.$
12. $1 + 4n - 10mn - 5m + 4n^2.$
13. $x^2 + 6xy - 2cx - 6cy + 9y^2.$
14. $x^2 - 12x - 14xy + 84y + 49y^2.$
15. $8y - 12x + 4y^2 + 9x^2 - 12xy.$
16. $a^2x^2 - 6abx + 11acx - 33bc + 9b^2.$
17. $a^4 + 4b^2 - 4a^2b + 10bc - 5a^2c.$
18. $6bc - 12ac - 4ab + 4a^2 + b^2.$
19. $p^2 + 4q^2 - 12pr + 4pq - 24qr.$
20. $9abd + 9d^2 - 6abc + 4c^2 - 12cd.$
21. $9b^2 - 39b + 42ab - 91a + 49a^2.$
22. $4b^2 + 4ab^2 - 10a^2b + 25a^2 - 20ab.$
23. $a^2m^2 + 5amx + 6am + 15x + 9.$
24. $9x^2 + 18ax + 12x + 12a + 4.$
25. $9y^2 + 25x^2 + 30xy - 12y - 20x.$

MISCELLANEOUS EXERCISES.

140. ***Factor the following:***

1. $a^4 - 81$.
2. $x^5 + y^5$.
3. $x^8 - 1$.
4. $m^5 - n^5$.
5. $x^6 - y^6$.
6. $a^3 - 125$.
7. $a^3 - 1$.
8. $x^3 + 64$.
9. $a^{12} - b^{12}$.
10. $x^5 + 32$.
11. $x^3 + 8n^3$.
12. $a^3 + 216$.
13. $x^3 + 1000$.
14. $x^4 - 100$.
15. $1 + x^7$.
16. $x^{15} - y^{15}$.
17. $x^3 + 343$.
18. $a^5 + 1$.
19. $x^4 - y^4$.
20. $a^{12} - 1$.
21. $a^2x^2 - b^2y^2$.
22. $(mn)^2 - (pq)^2$.
23. $x^4 - 625$.
24. $b^6 - 64$.
25. $x^5 + 243$.
26. $x^2 + 4xy + 4y^2$.
27. $a^2 - 6ab + 9b^2$.
28. $4a^2 - 12ax + 9x^2$.
29. $16m^2 - 40m + 25$.
30. $x^2 - 10x + 9$.
31. $x^2 - 34x + 225$.
32. $x^2 + 5x - 14$.
33. $x^4 - 26x^2 + 25$.
34. $a^2 + 6a + 8$.
35. $m^4 - 29m^2 + 100$.
36. $x^2 - 5x - 14$.
37. $16a^2 + 50a - 21$.
38. $3b^2 - 2b - 16$.
39. $3y^2 - 25y + 28$.
40. $2z^2 - 7z + 6$.
41. $5a^2 + 22a + 21$.
42. $16x^4 - 62x^2 + 21$.
43. $10p^2 - 41p + 39$.
44. $x^2 - 16x + 64$.
45. $10m^2 + 19m - 15$.
46. $6x^2 + x - 2$.

47. $2x^6 + x^3 - 6$.

48. $4x^4 + 13x^2 - 12$.

49. $4n^2 - 20n + 25$.

50. $49x^2y^2 + 154xy + 121$.

51. $a^2 - b^2 + 2bc - c^2$.

52. $a^2b - bx^2 + a^2x - x^3$.

53. $x^3 - 2xy + y^2 - 4z^2$.

54. $9x^2 + 20yz - 4y^2 - 25z^2$.

55. $ax^2 - ab^2 + b^2x - x^3$.

56. $(x^2 - yz)^2 - 16z^2$.

57. $(a^2 - b^2)^2 - (c^2 - d^2)^2$.

58. $ax - cy + cz + ay - cx - az$.

59. $9x^2 - 15xy - 6x + 5y + 1$.

60. $mx - nx + 2by + my - ny + 2bx$.

61. $6xy - 2mx + 9ny + 6nx + 4x^2 - 3my$.

62. $x^3 + 2x^2 - 19x - 20$.

63. $4y^2 - 6yz + 28y - 21z + 49$.

64. $x^3 - 10x^2 + 31x - 30$.

65. $x^3 - 9x^2 - 22x + 120$.

66. $2ay - a^2 - y^2 + b^2 + c^2 - 2bc$.

67. $x^3 - 4x^2 - 17x + 60$.

68. $16x^2 - 40xy - 24x + 30y + 25y^2$.

69. $x^3 + 6x^2 - 27x - 140$.

70. $a^3 + a^2b + a^2c - abc - b^2c - bc^2$.

71. $ab^2 - abx - axy + aby + b^2x + bxy - bx^2 - x^2y$.

72. $2mx - 4mr + 3nx - cx - 6nr + 2cr$.

73. $a^2 - b^2 - c^2 + d^2 - 2ad + 2bc$.

74. $a^2 + b^2c^2 + 5bcm + 2abc + 5am$.

75. $a^3 + 4a^2m + am^2 - 6m^3$.

76. $x^3 + 2x^2y - 23xy^2 - 60y^3$.

77. $ax + 2bx + cx - ay - 2by - cy$.

HIGHEST COMMON DIVISOR.

141. When two or more polynomials are arranged according to the descending powers of some common letter, the expression containing the highest powers of that letter, and which will exactly divide the given polynomials, is their greatest or highest common divisor. **(H. C. D.)**

142. The H. C. D. of two or more monomials is equal to the product of the greatest common divisor of their numerical co-efficients multiplied by the highest common divisor of their literal parts.

In algebra, as in arithmetic, there are two methods of finding the H. C. D. :

1st. By factoring.

2d. By division.

I. BY FACTORING.

143. While algebraic factors cannot be said to be prime, quantities may be separated into factors analogous to the prime factors of numbers.

Thus, of $a^4 - b^4$ such factors are $a - b$, $a + b$, and a^2+b^2; of $2ax^2 + 4axy + 2ay^2$ they are $2a$, $x+y$, and $x+y$.

I. Required to find the H. C. D. of $4a^3b^2c$ and $6a^2bc^3$.

Separating the quantities into their simple factors,

$$4a^3b^2c = 2 \times 2 \times a^3 \times b^2 \times c.$$
$$6a^2bc^3 = 2 \times 3 \times a^2 \times b \times c^3.$$
$$\text{H. C. D.} = 2 \times a^2 \times b \times c = 2a^2bc.$$

II. Required to find the highest common divisor of

$$x^2 + 7x + 10 \quad \text{and} \quad x^2 + 8x + 15.$$

Factoring the quantities,

$$x^2 + 7x + 10 = (x + 2)(x + 5),$$
$$x^2 + 8x + 15 = (x + 3)(x + 5).$$

Hence, $x + 5$ is the H. C. D.

Rule.—*Separate the quantities into their simple factors, and observe the factors common to all. The continued product of these common factors will be the* H. C. D.

EXERCISES.

144. ***Find the highest common divisor of:***

1. $9ax^2y^2$ and $15a^2xz$.
2. ab^2cu^2v and $3a^2buv^2w$.
3. $4ab^2$, $12a^2bx$, and $8ab^2y$.
4. $81x^3y^2z$ and $45abx^2y^4z^3$.
5. $a^2 + 2ab + b^2$ and $a^2 - b^2$.
6. $x^2 - 1$ and $x^2 - x - 2$.
7. $m^3 + n^3$ and $m^2 - n^2$.
8. $a^2 + 2a - 3$ and $a^2 + 5a + 6$.
9. $x^2 - 3x - 10$ and $x^2 - 8x + 15$.
10. $3x^3 - 3x^2 - 126x$ and $5x^2 - 55x + 140$.
11. $m^3 - m - 42$ and $m^2 + 15m + 54$.
12. $b^2 - b - 110$ and $b^2 - 16b + 55$.
13. $5ax^2 + 30ax + 25a$ and $5mx^3 - 5mx^2 - 150mx$.
14. $x^2 - 13x + 36$ and $x^2 + x - 90$.

15. $x^2 - 23x + 132$ and $x^2 - 4x - 77$.

16. $2x^2 - x - 15$ and $6x^2 + 13x - 5$.

17. $6x^2 - 19x + 10$ and $3x^2 - 14x + 8$.

18. $x^3 + 27$, $x^2 - 9$, and $2x^2 + 5x - 3$.

19. $x^2 + 4x + 4$, $x^3 + 8$, and $2x^2 + x - 6$.

20. $x^2 + 6x + 9$, $x^2 + x - 6$, and $3x^2 + 7x - 6$.

21. $x^3 - 125$, $x^2 - 10x + 25$, and $2x^2 - 11x + 5$.

22. $8y^3 - 216$, $4y^2 - 24y + 36$, and $10y^2 - 26y - 12$.

23. $a^3 - 27b^3$, $a^2 - 6ab + 9b^2$, and $2a^2 - ab - 15b^2$.

24. $3a^5 - 48a$, $2a^3b - 16b$, and $5a^2c - 20c$.

II. BY DIVISION.

145. The process of finding by division the highest common divisor of two or more quantities depends upon the following

Principles.—I. *A divisor of a quantity is a divisor of any multiple of that quantity.*

Thus, a divisor of A is a divisor of mA. For if A is equal to q times a divisor D, mA is equal to mq times that divisor.

II. *A common divisor of two quantities is a divisor of the sum or difference of any multiples of those quantities.*

Thus, if D is a common divisor of A and B, it is a divisor of $mA + nB$ and of $mA - nB$. For if A is equal to q times D, mA is equal to mq times D; and if B is equal to r times D, nB is equal to nr times D. Hence,

$$(mA + nB) \div D = mq + nr;$$

also,

$$(mA - nB) \div D = mq - nr.$$

146. To show how these principles are applied in finding the highest common divisor of two quantities, an illustration is first given, and then a demonstration.

I. ILLUSTRATION.

Process.

$$
\begin{array}{rl}
13D\,) & 33D\,(\,2 \\
 & \underline{26D} \\
 & 7D\,)\,13D\,(\,1 \\
 & \quad \underline{7D} \\
 & \quad 6D\,)\,7D\,(\,1 \\
 & \quad\quad \underline{6D} \\
 & \quad\quad D\,)\,6D\,(\,6 \\
 & \quad\quad\quad \underline{6D}
\end{array}
$$

Explanation. — Let there be two quantities equal respectively to $13D$ and $33D$. Since 13 and 33 are prime to each other, D is their highest common divisor. The process shows how the factor associated with D is decreased by dividing the greater by the less, the less by the first remainder, the first remainder by the second remainder, and so on. The factor associated with D finally becomes 1, and the highest common divisor appears as the last divisor.

II. DEMONSTRATION.

Process.

$$
\begin{array}{rl}
b\,) & a\,(\,m \\
 & \underline{mb} \\
 & c\,)\,b\,(\,n \\
 & \quad \underline{nc} \\
 & \quad d\,)\,c\,(\,r \\
 & \quad\quad \underline{rd}
\end{array}
$$

Explanation.—Let a and b represent two numbers of which a is the greater, or two algebraic expressions of which the highest exponent of the expression a is *not less* than the highest exponent of the expression b when both are arranged according to the descending powers of some leading letter. Divide a by b, and let the quotient be m and the remainder c. Divide b by c, and let the quotient be n and the remainder d. Divide c by d, and let the quotient be r and the remainder zero. Since d divides c, it will divide $nc + d$, which is equal to b. Since d divides b, it will divide $mb + c$, which is equal to a. Hence, ***d is a divisor of a and b.***

Again, every divisor of a and b is a divisor of b and $a - mb$, or c. Every divisor of b and c is a divisor of c and $b - nc$, or d. Hence, every divisor of a and b is a divisor of d. But no divisor of d can be greater or higher than d itself. Hence, ***d is the greatest or highest common divisor of a and b.***

I. Required to find the H. C. D. of

$$2x^3 - 7x - 2 \quad \text{and} \quad 2x^3 - x - 6.$$

$$\begin{array}{l}
2x^3 - x - 6 \;)\; 2x^3 - 7x - 2 \;(\; x + 1 \\
\qquad\qquad 2x^3 - x^2 - 6x \\
\qquad\qquad \overline{\qquad\qquad x^2 - \;\; x - 2} \\
\text{(Multiplying by 2.)} \quad 2 \\
\qquad\qquad \overline{\qquad\qquad 2x^2 - 2x - 4} \\
\qquad\qquad 2x^2 - \;\; x - 6 \\
\qquad\qquad \overline{\qquad\qquad\quad - x + 2} \\
\text{(Multiplying by } -1.) \qquad x - 2 \;)\; 2x^2 - x - 6 \;(\; 2x + 3 \\
\qquad\qquad\qquad\qquad\qquad 2x^2 - 4x \\
\qquad\qquad\qquad\qquad\qquad \overline{\qquad\quad 3x - 6} \\
\qquad\qquad\qquad\qquad\qquad 3x - 6 \\
\qquad\qquad\qquad\qquad\qquad \overline{\qquad\quad\quad}
\end{array}$$

Hence, the H. C. D. $= x - 2$.

II. Required to find the H. C. D. of

$$7x^3 - 23x + 6 \quad \text{and} \quad 5x^3 - 18x^2 + 11x - 6.$$

Multiplying by 7, to render divisible,

$$\begin{array}{l}
\qquad\qquad 5x^3 - 18x^2 + 11x - 6 \\
\qquad\qquad 7 \\
\qquad\qquad \overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\
7x^2 - 23x + 6 \;)\; 35x^3 - 126x^2 + 77x - 42 \;(\; 5x + 11 \\
\qquad\qquad 35x^3 - 115x^2 + 30x \\
\qquad\qquad \overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\
\text{(Multiplying} \qquad -11x^2 + 47x - 42 \\
\text{by } -7.) \qquad -7 \\
\qquad\qquad \overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\
\qquad\qquad 77x^2 - 329x + 294 \\
\qquad\qquad 77x^2 - 253x + \;\; 66 \\
\qquad\qquad \overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\
\qquad -76 \;)\; -76x + 228 \\
\text{(Rejecting the factor } -76.) \qquad x - 3 \;)\; 7x^2 - 23x + 6 \;(\; 7x - 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad 7x^2 - 21x \\
\qquad\qquad\qquad\qquad\qquad\qquad \overline{\qquad\quad -2x + 6} \\
\qquad\qquad\qquad\qquad\qquad\qquad -2x + 6 \\
\qquad\qquad\qquad\qquad\qquad\qquad \overline{\qquad\quad\quad}
\end{array}$$

Hence, the H. C. D. $= x - 3$.

Rule.—*Arrange the polynomials according to the descending powers of a common leading letter.*

Reject any factors discoverable by inspection, preserving those that are common to all as factors of the H. C. D.

Select as a dividend that polynomial the highest exponent of whose leading letter is equal to or greater than the highest exponent of the same letter in the other polynomial, and if its first term is not exactly divisible by the first term of the divisor, render it so by multiplication.

Divide the dividend by the divisor, then divide the first divisor by the remainder, the second divisor by the second remainder, and so continue until a divisor exactly contains a remainder. This last divisor, multiplied by the reserved common factors, if any, will be the H. C. D. *sought.*

If there are more polynomials, divide the third by this exact divisor, and so proceed until all have been employed.

NOTE.—Whenever possible, reject from each new divisor any factor not found in the dividend.

EXERCISES.

147. ***Find the H. C. D. of the following:***

1. $x^2 + 2x - 3$ and $x^2 + 5x + 6$.

2. $x^3 - 6x^2 + 11x - 6$ and $x^3 + 4x^2 + x - 6$.

3. $2x^2 + x - 3$ and $3x^2 - 4x + 1$.

4. $12x^2 - 88x + 128$ and $3x^3 - 33x^2 + 96x - 84$.

5. $2x^3 + 9x^2 + 4x - 15$ and $4x^3 + 8x^2 + 3x + 20$.

6. $x^4 - 9x^3 + 29x^2 - 39x + 18$ and $4x^3 - 27x^2 + 56x - 33$.

7. $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$.

8. $x^4 - 2x^3 + 4x^2 - 6x + 3$ and $x^4 - x^3 - 2x^2 + 3x - 1$.

9. $2x^6 - 96x^2 + 14$ and $7x^6 - 48x^4 + 1$.

10. $x^6 + x^4 + x^3 - x - 2$ and $2x^6 + x^5 - x^3 - x^2 - 1$.

11. $x^6 - 6x + 5$ and $x^6 + x^5 - 11x + 9$.

12. $x^6 - 2x^4 + 25x^3 - 39x + 14$ and $x^6 + 17x^3 - 13x + 2$.

13. $x^5 - 17x^3 + 7x^2 - 9$ and $x^5 - 16x^3 + 2x^2 + 5x - 12$.

14. $x^5 - 16x^3 + 2x^2 + 5x - 12$ and $2x^5 - 5x^4 - 12x^3 + 4x^2 - 9$.

15. $a^2x^5 + 83a^2x^2 - 48a^2x + 54a^2$ and $2mn^2x^5 - 48mn^2x^3 - 26mn^2x^2 - 36mn^2$.

16. $a^5b - 5a^3bc^2 + 9a^2bc^3 + 112abc^4 + 147bc^5$ and $a^4x - 5a^3c^2x - 12ac^3x + 28c^4x$.

17. $x^6 + 8x^3 + 9x^2 - 2$ and $x^6 - 3x^4 + 3x^2 - 1$.

18. $3x^5 - 46x^3 + 43x^2 - 36$ and $x^5 - 17x^3 + 7x^2 - 9$.

19. $x^6 - 8x^4 + 8x^2 - 1$ and $x^6 - 30x^3 + 32x^2 - 3$.

20. $x^5 - 5x^3 + 9x^2 + 112x + 147$ and $x^5 - 27x^3 - 77x - 245$.

21. $2x^5 - 10x^3 - 38x^2 - 98$ and $3x^5 + 36x^2 + 237x + 84$.

22. $2abx^5 - 164abx - 6ab$ and $3m^2nx^5 + 84m^2nx^2 - 27m^2n$.

LOWEST COMMON MULTIPLE.

148. When two or more polynomials are arranged according to the descending powers of some common letter, the expression containing the lowest exponents of that letter, and which is exactly divisible by each of the given polynomials, is their lowest common multiple. **(L. C. M.)**

The L. C. M. of two or more monomials is equal to the continued product of the highest powers of all their different factors.

149. To find the L. C. M. of two or more quantities that can be factored by inspection.

I. Required to find the L. C. M. of $8a^2bc$ and $21a^3b^2c^2$.

$$\begin{aligned} 8a^2bc &= 2^3 \times a^2 \times b \times c \\ 21a^3b^2c^2 &= 3 \times 7 \times a^3 \times b^2 \times c^2 \\ \text{L. C. M.} &= 2^3 \times 3 \times 7 \times a^3 \times b^2 \times c^2 = 168a^3b^2c^2. \end{aligned}$$

Explanation.—The L. C. M. of the quantities must contain each of their factors with its highest exponent. These are 2^3, 3, 7, a^3, b^2, and c^2, and their continued product is the L. C. M. required.

II. Required to find the L. C. M. of $a^2 - b^2$, $a^2 + 2ab + b^2$, and $a^2 - 2ab + b^2$.

$$\begin{aligned} a^2 - b^2 &= (a + b)(a - b) \\ a^2 + 2ab + b^2 &= (a + b)(a + b) \\ a^2 - 2ab + b^2 &= (a - b)(a - b) \\ \text{L. C. M.} &= (a + b)^2(a - b)^2 = a^4 - 2a^2b^2 + b^4. \end{aligned}$$

150. To find the L. C. M. of two or more quantities that cannot easily be factored by inspection.

III. Required to find the L. C. M. of $x^3 - 5x^2 - 17x + 21$, $x^3 - 2x^2 - 29x - 42$, and $x^3 - 11x^2 + 31x - 21$.

Process.

$$\left.\begin{matrix} x^3 - 5x^2 - 17x + 21 \\ x^3 - 2x^2 - 29x - 42 \\ x^3 - 11x^2 + 31x - 21 \end{matrix}\right\} \div (x - 7) = \left\{\begin{matrix} (x-1)(x+3) \\ (x+2)(x+3) \\ (x-3)(x-1) \end{matrix}\right.$$

$$\text{L. C. M.} = (x-7)(x-1)(x+2)(x+3)(x-3)$$
$$= x^5 - 6x^4 - 18x^3 + 68x^2 + 81x - 126.$$

Explanation.—The H. C. D. of the quantities, found by the process already explained, is $x - 7$. Dividing the quantities in order by $x - 7$, and factoring the quotients, gives the binomials shown above. These pairs of binomials, each with the H. C. D., $x - 7$, are respectively the factors of the quantities. From these factors the L. C. M. is written, as already explained.

Rule.—I. *Resolve the quantities into their simple factors; take each factor the greatest number of times it is found in any of the quantities; the product of these factors will be the* L. C. M.

II. *If the simple factors of the quantities cannot be found by inspection, first ascertain their* H. C. D., *and divide each quantity by it. Then find the* L. C. M. *of these quotients, and multiply it by the* H. C. D. *of the quantities; the result will be their* L. C. M.

NOTE.—The L. C. M. of two or more quantities that have no common divisor is their product.

Thus, the L. C. M. of $a + b$, $a - b$, and $a^2 + b^2$ is

$$(a + b)(a - b)(a^2 + b^2) = a^4 - b^4.$$

EXERCISES.

151. ***Find the L. C. M. of the following:***

1. $4a^2b^3$ and $3a^4b$.

2. $9a^3b^7$ and $15a^4b^6$.

3. $15x^3y^2z$ and $21x^2y^3z^2$.

4. $6x^2yz^2$ and $14x^3y^2z$.

5. $2x + 1$ and $2(4x^2 - 1)$.

6. $x^3 - 8$ and $x^2 - 4$.

7. $6(x^2y - xy^2)$ and $16(x^3 - y^3)$.

8. $x^2 + 2x - 35$ and $x^2 + x - 42$.

9. $x^2 - 3x - 4$ and $x^2 - x - 12$.

10. $x^2 + 2x - 99$ and $x^2 - x - 132$.

11. $x^2 - 3x - 28$, $x^2 + x - 12$, and $x^2 - 10x + 21$.

12. $x^3 - x^2 + x - 1$ and $x^3 - 7x + 6$.

13. $12x^2 - x - 1$ and $6x^2 - 5x + 1$.

14. $3x^2 - 5x + 2$ and $4x^3 - 4x^2 - x + 1$.

15. $a^3 + 6a + 7$ and $a^3 - 2a^2 + 8a - 7$.

16. $15(a^2b - ab^2)$, $21(a^3 - ab^2)$, and $35(ab^2 + b^3)$.

17. $x^2 - 1$, $x^2 + 1$, $x^4 + 1$, and $x^8 - 1$.

18. $x^3 + 3x^2 - x - 3$ and $x^3 + 4x^2 + x - 6$.

19. $x^3 - 7x - 6$ and $x^3 + x^2 - 9x - 9$.

20. $x^4 - 7x^3 - 7x^2 + 43x + 42$ and $x^4 - 9x^3 + 9x^2 + 41x - 42$.

21. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$, and $x^3 - 8x^2 + 19x - 12$.

FRACTIONS.

152. A **fraction** is an expression of division in which the dividend is written above, and the divisor below, a horizontal line.

Thus, $\frac{a}{b}$, $\frac{ax}{y}$, $\frac{4x}{5}$ or $\frac{4}{5}x$, $\frac{mn}{pq}$, $\frac{a+b}{xy}$, $\frac{x-y}{m+n}$.

153. The **numerator** is the quantity above the line. The **denominator** is the quantity below the line.

154. The **terms** of a fraction are its numerator and denominator.

REDUCTION OF FRACTIONS.

155. To reduce entire or mixed quantities to the fractional form.

156. *Any algebraic expression may be written as a fraction having 1 for its denominator.*

Thus, $a = \frac{a}{1}$; $xy = \frac{xy}{1}$, $a + b = \frac{a+b}{1}$.

157. *Both terms of a fraction may be multiplied by the same quantity without changing the value of the fraction.*

Thus, $\frac{a}{1} \times \frac{b}{b} = \frac{ab}{b}$; but $ab \div b = a$, the value of $\frac{a}{1}$.

I. Required to reduce $5a + 3b$ to an equivalent fraction whose denominator is $5a - 3b$.

Explanation.—The given quantity is first written in the form of a fraction whose denominator is 1. Multiplying both terms of this fraction by the proposed denominator changes its form but not its value. The result is the form required.

Process.

$$5a + 3b = \frac{5a + 3b}{1};$$

$$\frac{5a + 3b}{1} \times \frac{5a - 3b}{5a - 3b} = \frac{25a^2 - 9b^2}{5a - 3b}$$

II. Required to change $3a - \frac{b - a}{2}$ to fractional form.

Explanation.—The entire part, $3a$, is first reduced to an equivalent fraction having the same denominator as the fractional part. The entire part and the fractional part are now expressed as fractions with equal denominators, and may be written as one fraction and then simplified in accordance with the principle explained in Art. **69**.

Process.

$$\frac{3a}{1} \times \frac{2}{2} = \frac{6a}{2};$$

$$\frac{6a}{2} - \frac{b - a}{2} = \frac{6a - (b - a)}{2}$$

$$= \frac{7a - b}{2}.$$

Rule.—I. For Entire Forms.—*Multiply the entire quantity by the required denominator, and write the product over that denominator.*

II. For Mixed Forms.—*Multiply the entire part by the denominator of the fractional part; to the product annex the numerator of the fractional part with its sign; under the result write the denominator of the fractional part. Unite the terms of the numerator, if possible.*

EXERCISES.

158. ***Reduce to fractional form:***

1. $3a + \frac{x}{y}$.
2. $2 + \frac{a}{x}$.
3. $3a + \frac{ab - a}{b}$.
4. $3a^2x - \frac{a^2x^2 - a^3}{x}$.
5. $1 + \frac{x}{1 + x}$.
6. $2a + \frac{4b^2}{a - b}$.
7. $a + x + \frac{a^2 + x^2}{a - x}$.
8. $3 + \frac{3}{x^2 - 1}$.
9. $ax + 4 - \frac{ax^2 + 4x}{a + x}$.
10. $\frac{5bx}{a^2} - 2x$.
11. $\frac{a + b + c}{a + c} - 1$.
12. $3a - b - \frac{2a^2 - b}{a + 1}$.
13. $x^3 - 1 + \frac{x + y}{x - y}$.
14. $a + 2b + \frac{ab + 5b^2}{a - 3b}$.
15. $a(m + n) + \frac{an^2}{m - n}$.
16. $a - x + \frac{a^2 + x^2 - 5}{a + x}$.
17. $x^2 - xy + y^2 - \frac{x^4 + y^4}{x^2 + xy + y^2}$.
18. $x^2 + 2x - 1 - \frac{7x^2 + 9x}{x + 5}$.
19. $x^3 + 3x - 5 - \frac{5x^3 - 13x}{x^2 + 2x - 1}$.
20. $x^3 - x - 1 - \frac{x^4 - 2x^2 - 3x + 2}{x^2 + x + 1}$.

21. Change $3a^4xy^2$ to an equivalent fraction having $2a^2x^3$ for its denominator.

22. $a^2 - x^2$ to a fraction whose denominator is $a^2 + x^2$.

23. $5x^2y^3z$ to a fraction whose denominator is $4axz$.

24. $ax(x + y)$ to a fraction whose denominator is $x - y$.

25. $a^3 + a^2b + ab^2 + b^3$ to a fraction whose denominator is $a - b$.

159. To reduce fractions to entire or mixed quantities.

I. Required to reduce $\frac{17a^2}{5}$ to a mixed form.

Explanation.—Dividing $17a^2$ by 5, as in the division of monomials, gives the form required.

Process.

$$\frac{17a^2}{5} = \frac{15a^2}{5} + \frac{2a^2}{5} = 3a^2 + \frac{2a^2}{5}.$$

II. Required to reduce $\frac{a^3 - 2ab + b^3}{a + b}$ to a mixed form.

Explanation.—Dividing the numerator by the denominator gives $a^2 - ab + b^2$, with a remainder $-2ab$. Adding the remainder to the quotient gives the form required.

Process.

$$(a^3 - 2ab + b^3) \div (a + b) =$$
$$a^2 - ab + b^2 + \frac{-2ab}{a + b}.$$

Rule.—*Perform the division indicated by the fraction. If there is a remainder, write it over the denominator, and to the entire part of the quotient annex the fraction thus formed, placing before it the sign* $+$.

NOTE.—When the remainder is a single term and negative, the sign $-$ may be written before the fraction and omitted from the numerator. Thus, in the foregoing example, $a^2 - ab + b^2 + \frac{-2ab}{a + b}$ may be written $a^2 - ab + b^2 - \frac{2ab}{a + b}$. See Arts. **168** and **169**.

EXERCISES.

160. ***Reduce to integral or mixed forms:***

1. $\frac{23a}{9}$.
2. $\frac{15abc}{4}$.
3. $\frac{36ac + 4c}{9}$.
4. $\frac{8a^2 + 3b}{4a}$.
5. $\frac{12x^2 - 5y}{6x}$.
6. $\frac{2a^2x - x^3}{a}$.
7. $\frac{x^2 + 3x + 2}{x + 3}$.
8. $\frac{2x^2 - 6x - 1}{x - 3}$.
9. $\frac{x^2 - 4xy + 4y^2}{x - 2y}$.

10. $\dfrac{x^3+y^3}{x-y}$.

11. $\dfrac{a^4+b^4}{a^2+b^2}$.

12. $\dfrac{a^3-b^3}{a+b}$.

13. $\dfrac{x^2-37}{x+7}$.

14. $\dfrac{x^4-1}{x+1}$.

15. $\dfrac{a^3-a^2b+ab^2-b^3}{a-b}$.

16. $\dfrac{x^5-x^4+2x+1}{x-1}$.

17. $\dfrac{x^5-5x^3+9x^2+x-51}{x^3-4x^2+4x-7}$.

18. $\dfrac{2x^5-5x^4-12x^3+5x^2-9}{x^3-5x^2+5x-3}$.

161. To reduce a fraction to its simplest form.

162. A fraction is in its **simplest form** when its terms are not themselves fractional, and have no common factor.

163. *Both terms of a fraction may be divided by the same quantity without changing the value of the fraction.*

Thus, $\dfrac{ab \div b}{b \div b} = \dfrac{a}{1}$; but $\dfrac{ab}{b} = ab \div b = a = \dfrac{a}{1}$.

Hence, *Factors common to both terms of a fraction may be omitted without affecting the value of the fraction.*

I. Required to simplify $\dfrac{15a^3xy^2}{25a^2x^2y^3}$.

Explanation. — Separating each term of the fraction into two factors, one of which is their highest common divisor, and omitting this common factor, gives the simplest form of the fraction.

Process.

$$\frac{15a^3xy^2}{25a^2x^2y^3} = \frac{5a^2xy^2 \times 3a}{5a^2xy^2 \times 5xy} = \frac{3a}{5xy}.$$

II. Required to simplify $\frac{a^2 - b^2}{a^4 - b^4}$.

Process.

$$\frac{a^2 - b^2}{a^4 - b^4} = \frac{a^2 - b^2}{(a^2 - b^2)(a^2 + b^2)} = \frac{1}{a^2 + b^2}.$$

Explanation.—The denominator may be separated into two factors, of which one is equal to the numerator, $a^2 - b^2$. Omitting this common factor gives the simplest form required.

III. Required to simplify $\frac{x^3 - 2x^2 - 14x + 3}{x^3 - 24x + 5}$.

Process.

$$\frac{(x^3 - 2x^2 - 14x + 3) \div (x^2 - 5x + 1)}{(x^3 - 24x + 5) \div (x^2 - 5x + 1)} = \frac{x + 3}{x + 5}.$$

Explanation.—Since the terms of the fraction are not easily factored by inspection, their H. C. D. is found by the method of division. Dividing both terms by their H. C. D., $x^2 - 5x + 1$, gives the simplest form required.

Rule.—I. *Resolve the terms of the fraction into their simple factors, and omit those that are common*; or,

II. *Divide the terms by their highest common divisor.*

EXERCISES.

164. *Simplify:*

1. $\frac{3x^2y^2}{9x^3y^4}$.
2. $\frac{14xy^2z^3}{49x^2yz^2}$.
3. $\frac{75x^5y^2z}{100x^3y^4z^2}$.
4. $\frac{35m^4n^2xz^2}{49pqm^3nzy}$.
5. $\frac{16a^4b^2}{72a^3bcde}$.
6. $\frac{6a^2x^3}{9a^3x^2}$.
7. $\frac{12x^4y^5z^3}{15x^5y^3z^2u^2}$.
8. $\frac{17a^2bc^3}{85a^3c^2x}$.
9. $\frac{13abc^2xy^5}{65a^2b^2cx^4y}$.

10. $\frac{a^2 + ab}{2ab}$.

11. $\frac{a^2 + ab}{a^2 - ab}$.

12. $\frac{5a^2x}{15a^2x - 10ay^2}$.

13. $\frac{ax + bx}{a^2 - b^2}$.

14. $\frac{3a^2 + 3ax}{a^2 - x^2}$.

15. $\frac{(a + b)^2}{a^2 - b^2}$.

16. $\frac{a^3 + b^3}{a^2 - b^2}$.

17. $\frac{x^3 - xy^2}{x^4 - y^4}$.

18. $\frac{x^2 + 3x + 2}{x^2 + 6x + 5}$.

19. $\frac{10a^2x}{5a^2x - 15ay^2}$.

20. $\frac{2(x + y)^2}{3(x^2 - y^2)}$.

21. $\frac{2x^2 + x - 15}{2x^2 - 19x + 35}$.

22. $\frac{x^2 + 10x + 21}{x^2 - 2x - 15}$.

23. $\frac{x^2 + 2x - 3}{x^2 + 4x + 3}$.

24. $\frac{x^4 - x^2y^2}{x^4 - y^4}$.

25. $\frac{x^5 + x^3}{x^4 - 1}$.

26. $\frac{x^2 - 7x + 10}{x^2 - 2x - 15}$.

27. $\frac{x^3 + 5x + 6}{x^3 + x + 10}$.

28. $\frac{3x^2 + 23x - 36}{4x^2 + 33x - 27}$.

29. $\frac{a + 3b}{17a^2 - 153b^2}$.

30. $\frac{x^3 - ax^2}{x^2 - 2ax + a^2}$.

31. $\frac{axy}{bx^2y + cxy^2}$.

32. $\frac{3mn + 3mr}{n^2 - r^2}$.

33. $\frac{1 + x + x^2}{1 - x^3}$.

34. $\frac{6x^2 + x - 15}{6x^2 - 11x - 35}$.

35. $\frac{6m^2 + mx - 15x^2}{15m^2 + 16mx - 15x^2}$.

36. $\frac{6x^2 - 11x - 10}{6x^2 - 19x + 10}$.

37. $\frac{2x^2 + 3x + 1}{x^2 - x - 2}$.

38. $\dfrac{x^3 + 2x^2 - 5x - 6}{x^3 - 3x^2 - 10x + 24}$.

39. $\dfrac{x^3 - 57x - 56}{x^3 + x^2 - 64x - 64}$.

40. $\dfrac{9x^3 - 46x + 35}{15x^3 + 46x^2 - 49}$.

41. $\dfrac{4x^3 - 23x + 5}{10x^3 - 23x^2 + 1}$.

42. $\dfrac{x^4 - y^4}{x^3 + x^2y^2 + x^2y + y^3}$.

43. $\dfrac{x^3 + x^2 + x + 1}{x^4 - 1}$.

165. To reduce two or more fractions to equivalent fractions having the lowest common denominator.

I. Required to change $\frac{a}{bc^2}$, $\frac{c}{bd}$, and $\frac{m}{b^2cd}$ to their lowest common denominator.

Process.

L. C. M. of bc^2, bd, and $b^2cd = b^2c^2d$.

$b^2c^2d \div bc^2 = bd$; $\quad \dfrac{a \times bd}{bc^2 \times bd} = \dfrac{abd}{b^2c^2d}$.

$b^2c^2d \div bd = bc^2$; $\quad \dfrac{c \times bc^2}{bd \times bc^2} = \dfrac{bc^3}{b^2c^2d}$.

$b^2c^2d \div b^2cd = c$; $\quad \dfrac{m \times c}{b^2cd \times c} = \dfrac{cm}{b^2c^2d}$.

Explanation.—A common denominator of two or more fractions is a common multiple of their denominators, and the L. C. D. of two or more fractions is in like manner the L. C. M. of their denominators. The L. C. M. of the denominators bc^2, bd, and b^2cd is b^2c^2d. Dividing this by the denominator of each fraction, multiplying its numerator by the quotient and writing the product over the L. C. M., gives the fractions required.

Rule.—*Find the* L. C. M. *of the denominators. Divide it by each denominator. Multiply the numerator of each fraction by the quotient obtained by thus dividing by its denominator, and write each product over the* L. C. M. *of the denominators.*

EXAMPLES.

166. ***Reduce the following fractions to equivalent fractions having the L. C. D.:***

1. $\frac{3}{4x}, \frac{4}{6x^2}, \frac{5}{12x^3}$.

2. $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}$.

3. $\frac{1}{ab}, \frac{2}{ac}, \frac{3}{bc}$.

4. $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$.

5. $\frac{3a}{4x^2}, \frac{6ax}{8xy^2}, \frac{5b}{x^2y^2}$.

6. $\frac{a-x}{a+x}, \frac{a+x}{a-x}$.

7. $\frac{a-b}{a+b}, \frac{a+2b}{a-b}, \frac{a^2}{a^2-b^2}$.

8. $\frac{a-b}{a^2+ab+b^2}, \frac{a^2+b^2}{a^3-b^3}, \frac{b}{a-b}$.

9. $\frac{3}{1+x}, \frac{5}{4+4x}, \frac{2x}{1-x^2}$.

10. $\frac{x}{x-y}, \frac{y}{x+y}, \frac{xy}{x^2-y^2}, \frac{y^2}{x^2+y^2}$.

11. $\frac{a}{a^2-ab+b^2}, \frac{b}{a^2+ab+b^2}, \frac{c}{a^4+a^2b^2+b^4}$.

12. $\frac{x-1}{x^2-8x+15}, \frac{x+3}{x^2-4x-5}, \frac{x-5}{x^2+4x+3}$.

13. $\frac{x-3}{x^2-4}, \frac{x-2}{x^2+x-6}, \frac{x^2+4}{x^2+6x+9}$.

14. $\frac{5}{x^2+x-20}, \frac{3}{x^2-x-12}, \frac{2}{x^2+x-6}$.

15. $\frac{b+x}{(a-x)(a-y)}, \frac{a+x}{(a-y)(b-x)}$,

ADDITION AND SUBTRACTION.

167. Since the addition of a negative quantity is equivalent to the subtraction of an equal positive quantity, it follows that algebraic addition and subtraction are substantially the same process.

168. To add or subtract algebraic fractions.

Operations in addition and subtraction of fractions are often much simplified by first making changes in the signs of the fractions to be operated upon. To show how this may be done without affecting the value of the fraction,

Let $\frac{a}{b}$ be a fraction such that a contains b exactly m times. Then, by applying the principles governing the signs in addition, subtraction, and division,

1. $+\frac{+a}{+b} = +(+m) = +m.$

2. $+\frac{-a}{-b} = +(+m) = +m.$

3. $-\frac{+a}{+b} = -(+m) = -m.$

4. $-\frac{-a}{-b} = -(+m) = -m.$

5. $+\frac{-a}{+b} = +(-m) = -m.$

6. $+\frac{+a}{-b} = +(-m) = -m.$

7. $-\frac{-a}{+b} = -(-m) = +m.$

8. $-\frac{+a}{-b} = -(-m) = +m.$

Hence,

1. *Changing one sign changes the sign of the value of the fraction.*

2. *Changing two signs does not affect the value of the fraction.*

3. *Changing three signs changes the sign of the value of the fraction.*

169. To apply these principles when the terms of the fraction are polynomials, *changing all the signs of a polynomial term counts as one change of sign.*

Thus, $\frac{a-b}{c-d} = \frac{-a+b}{-c+d}$, which is usually written $\frac{b-a}{d-c}$.

Also, $\frac{a-b}{c-d} = -\frac{b-a}{c-d}$; and $\frac{a-b}{c-d} = -\frac{a-b}{d-c}$.

Again, $\frac{a^2-b^2}{c^2-d^2} = \frac{b^2-a^2}{d^2-c^2} = -\frac{b^2-a^2}{c^2-d^2} = -\frac{a^2-b^2}{d^2-c^2}$.

Changing the signs of both terms of a fraction is equivalent to multiplying both terms by -1.

Thus, $\frac{(a-b)\times -1}{(c-d)\times -1} = \frac{-a+b}{-c+d}$, or $\frac{b-a}{d-c}$.

170. A fraction whose numerator is equal to 0 is itself equal to 0.

Thus, $\frac{3}{5}-\frac{3}{5} = \frac{3-3}{5} = \frac{0}{5}$; but $\frac{3}{5}-\frac{3}{5} = 0$.

Again, $\frac{a}{b}-\frac{a}{b} = \frac{a-a}{b} = \frac{0}{b}$; but $\frac{a}{b}-\frac{a}{b} = 0$.

I. Required to find the value of $\frac{a}{bd} + \frac{c}{d} + \frac{x}{d^2}$.

$$\text{L. C. D.} = bd^2.$$

$$bd^2 \div \left\{ \begin{array}{lll} bd = d; & \frac{a \times d}{bd \times d} & = \frac{ad}{bd^2} \\ d = bd; & \frac{c \times bd}{d \times bd} & = \frac{bcd}{bd^2} \\ d^2 = b; & \frac{x \times b}{d^2 \times b} & = \frac{bx}{bd^2} \end{array} \right\} = \frac{ad + bcd + bx}{bd^2}, \textit{ Ans.}$$

II. Required to find the value of $\frac{3x - 2y}{12} + \frac{5y - 4x}{18}$.

$$\text{L. C. D.} = 36; \quad 36 \div \left\{ \begin{array}{l} 12 = 3; \\ 18 = 2; \end{array} \right.$$

$$\left. \begin{array}{lll} \frac{(3x - 2y) \times 3}{12 \times 3} & = & \frac{9x - 6y}{36} \\ \frac{(5y - 4x) \times 2}{18 \times 2} & = & \frac{-8x + 10y}{36} \end{array} \right\} = \frac{x + 4y}{36}, \textit{ Ans.}$$

III. Required to simplify $\frac{ax}{a - b} + \frac{bx}{a + b} + \frac{2abx}{b^2 - a^2}$.

In this example, $\frac{2abx}{b^2 - a^2}$ is the same as $\frac{-2abx}{a^2 - b^2}$.

$$\text{L. C. D.} = a^2 - b^2; \qquad (a^2 - b^2) \div \left\{ \begin{array}{l} (a - b) = (a + b); \\ (a + b) = (a - b); \\ (a^2 - b^2) = 1; \end{array} \right.$$

$$\left. \begin{array}{lll} \frac{ax \times (a + b)}{(a - b) \times (a + b)} & = & \frac{a^2x + abx}{a^2 - b^2} \\ \frac{bx \times (a - b)}{(a + b) \times (a - b)} & = & \frac{abx - b^2x}{a^2 - b^2} \\ \frac{-2abx \times 1}{(a^2 - b^2) \times 1} & = & \frac{-2abx}{a^2 - b^2} \end{array} \right\} = \frac{x(a^2 - b^2)}{a^2 - b^2} = x, \textit{ Ans.}$$

IV. Simplify $\frac{3x+5}{2-x} - \frac{3-5x}{2+x} + \frac{x(24-x)}{x^2-4}$.

$$-\frac{3-5x}{2+x} = +\frac{5x-3}{2+x}, \quad \text{and} \quad \frac{x(24-x)}{x^2-4} = \frac{-24x+x^2}{4-x^2}.$$

$$\text{L. C. D.} = 4-x^2; \quad (4-x^2) \div \left\{ \begin{matrix} (2-x) = 2+x \\ (2+x) = 2-x \\ (4-x^2) = 1 \end{matrix} \right\}$$

$$\left. \begin{matrix} (3x+5)(2+x) = 10+11x+3x^2 \\ (5x-3)(2-x) = -6+13x-5x^2 \\ (-24x+x^2) \times 1 = -24x+x^2 \end{matrix} \right\} = \frac{4-x^2}{4-x^2} = 1, \textit{ Ans.}$$

171. When the denominators consist of two or more factors enclosed in parentheses, signs may often be changed so as to simplify the operation. The following examples will show how these changes may be made without affecting the value of the fractions.

$$(a-b)(c-d) = ac - bc - ad + bd.$$

$$(-a+b)(c-d) = -ac + bc + ad - bd, \text{ one factor changed changes sign of result.}$$

$$(-a+b)(-c+d) = ac - bc - ad + bd, \text{ two factors changed does not affect the result.}$$

Again,
$$(x-y)(x+y)(x^2+y^2) = x^4 - y^4.$$
$$(y-x)(x+y)(x^2+y^2) = y^4 - x^4.$$
$$(y-x)(-x-y)(x^2+y^2) = x^4 - y^4.$$
$$(y-x)(-x-y)(-x^2-y^2) = y^4 - x^4.$$

V. Find the sum of

$$\frac{1}{(x-3)(x-4)} + \frac{2}{(x-2)(4-x)} + \frac{1}{(2-x)(3-x)}.$$

As the example stands, there seem to be no like factors in the denominators. It may, however, be written

$$\frac{1}{(x-3)(x-4)} + \frac{-2}{(x-2)(x-4)} + \frac{1}{(x-2)(x-3)}.$$

The L. C. D. is now easily seen to be

$$(x-2)(x-3)(x-4).$$

Dividing the L. C. D. by the denominators in order gives respectively $x-2$, $x-3$, $x-4$. Whence,

$(x-2) + (x-3) \times (-2) + (x-4) = 0$, the sum of the numerators.

Hence, the result is $\frac{0}{(x-2)(x-3)(x-4)} = 0.$

Rule.—*Reduce the fractions to equivalent fractions having the lowest common denominator. Add the numerators of these equivalent fractions and write the sum over their common denominator. Simplify the result, if possible.*

EXERCISES.

172. ***Perform the operations indicated:***

1. $\frac{a}{x} + \frac{b}{x} + \frac{c}{x^2}.$

2. $\frac{a+b}{6} + \frac{a-b}{9}.$

3. $\frac{x^2+y^2}{9} + \frac{x^2-y^2}{12}.$

4. $\frac{5+x}{y} + \frac{3-ax}{ay} + \frac{b}{3a}.$

5. $\frac{3}{4a} - \frac{5}{2x}$.

6. $\frac{3a - 5b}{4} + \frac{4b - 2a}{3}$.

7. $\frac{1}{a - b} + \frac{1}{a + b}$.

8. $\frac{7}{x + y} + \frac{5}{x - y}$.

9. $\frac{ax}{b^2c} - \frac{p}{b^2n^2}$.

10. $\frac{am + b}{m^2 - n^2} - \frac{a}{m + n}$.

11. $\frac{x + y}{x - y} - \frac{x - y}{x + y}$.

12. $\frac{a + n}{x + y} + \frac{a - n}{x - y}$.

13. $\frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}$.

14. $\frac{1}{m - 1} - \frac{m}{m^2 - 1}$.

15. $\frac{3 - 8x}{x^2 - 9} - \frac{x^2 + 3x + 1}{x + 3}$.

16. $\frac{8}{x + 5} + \frac{65 - 3x}{x^2 - 25}$.

17. $\frac{1 + 3x}{1 - 3x} - \frac{1 - 3x}{1 + 3x}$.

18. $ax + \frac{1}{bx} + \frac{2}{cx^2}$.

19. $\frac{a^2}{8x^3} + \frac{a^3b}{12x} + \frac{a^4b^2y}{20mx}$.

20. $\frac{xy}{x - y} + \frac{x^3 + y^3}{x^2 - y^2}$.

21. $\frac{x}{x - y} - \frac{y}{y - x}$.

22. $\frac{1}{x - a} + \frac{1}{x + a} - \frac{2}{x}$.

23. $\frac{a}{a - x} + \frac{3a}{a + x} - \frac{2ax}{a^2 - x^2}$.

24. $\frac{n(4y + 1)}{x^2 - y^2} + \frac{m + 2n}{x + y} + \frac{m - 2n}{x - y}$.

25. $\frac{1 + x}{x + 3} + \frac{1 - x}{x - 2} + \frac{2x - 1}{x + 2}$.

26. $\frac{a - 2b}{3c} - \frac{b - 3c}{2a} + \frac{4ab + 3bc}{6ac}$.

27. $\frac{3a+7}{13} - \frac{a-3}{3} + \frac{2(2a+9)}{39}$.

28. $\frac{2m}{m^2-n^2} + \frac{1}{m+n} - \frac{1}{m-n}$.

29. $\frac{1}{a+b} + \frac{b}{a^2-b^2} - \frac{a}{a^2+b^2}$.

30. $\frac{2}{x} - \frac{3}{2x-1} - \frac{2x-3}{4x^2-1}$.

31. $\frac{2b-a}{x-b} + \frac{b-2a}{x+b} + \frac{3x(a-b)}{x^2-b^2}$.

32. $\frac{3}{x} - \frac{5}{2x-1} - \frac{2x-7}{4x^2-1}$.

33. $\frac{1}{x+1} - \frac{2}{x+2} + \frac{1}{x+3}$.

34. $\frac{3}{2x-4} - \frac{1}{x+2} - \frac{x+10}{2x^2+8}$.

35. $\frac{2}{x+4} - \frac{x-3}{x^2-4x+16} + \frac{x^2}{x^3+64}$.

36. $\frac{a(2a-b)}{a^3+b^3} - \frac{a-b}{a^2-ab+b^2}$.

37. $\frac{a^4+3a^2b^2+b^4}{a^3-b^3} - \frac{a^2-ab+b^2}{a-b}$.

38. $\frac{2x+14}{x^3+x^2-49x-49} - \frac{1-x}{x^2-6x-7}$.

39. $\frac{2x^2}{x^4-y^4} - \frac{x+y}{x^3+x^2y+xy^2+y^3}$.

40. $\dfrac{1}{x-1} - \dfrac{1}{x+2} - \dfrac{3}{(x+2)^2}$.

41. $\dfrac{(a^2+b^2)^2}{ab\,(a-b)^2} - \dfrac{a}{b} - \dfrac{b}{a} - 2$.

42. $\dfrac{x}{x-y} + \dfrac{3x}{x+y} + \dfrac{2xy}{y^2-x^2}$.

43. $\dfrac{a}{b-a} - \dfrac{3a}{a+b} - \dfrac{2ab}{a^2-b^2}$.

44. $\dfrac{a^2}{(a-b)\,(a-c)} + \dfrac{b^2}{(b-a)\,(b-c)} + \dfrac{c^2}{(c-a)\,(c-$

45. $\dfrac{3}{1-2x} - \dfrac{7}{1+2x} - \dfrac{4-20x}{4x^2-1}$.

46. $\dfrac{c}{(x-a)\,(a-b)} - \dfrac{c}{(b-x)\,(b-a)}$.

47. $\left\{\dfrac{1}{x^3} + \dfrac{1}{x^2} + \dfrac{x-1}{x^2+1}\right\} - \left\{\dfrac{1}{x} + \dfrac{1}{(x^2+1)^2}\right\}$.

48. $\dfrac{5}{2\,(x+1)} - \dfrac{1}{10\,(x-1)} - \dfrac{24}{5\,(2x+3)}$.

49. $\dfrac{2}{a-b} + \dfrac{2}{b-c} - \dfrac{2}{a-c} +$
$\dfrac{(a-b)^2+(b-c)^2+(c-a$
$(a-b)\,(c-b)\,(a-c)$

50. $\dfrac{a^3+b^3}{ab\,(a+b)} - \left\{\dfrac{a}{b} + \dfrac{b}{a} + 2\right\}$.

51. $\dfrac{1}{(x-3)\,(x-4)} + \dfrac{2}{(4-x)\,(x-2)} + \dfrac{1}{(2-x)\,(3-x$

52. $\dfrac{1}{a-2b} - \dfrac{4}{a-b} + \dfrac{6}{a} - \dfrac{4}{a+b} + \dfrac{1}{a+2b}$.

MULTIPLICATION.

173. Since an entire or a mixed quantity may be expressed in fractional form, multiplication of fractions consists of but one operation ; that is,

174. To find the product of two or more fractions.

I. Required to multiply $\frac{a}{b}$ by c. Also $\frac{a}{b}$ by $\frac{m}{n}$.

$$\frac{a}{b} \times c = \frac{a}{b} \times \frac{c}{1} = \frac{ac}{b}, \text{ Ans.}$$

$$\frac{a}{b} \times \frac{m}{n} = \frac{am}{bn}, \text{ Ans.}$$

II. Required to find the product of $\frac{a^2b}{m^2x^3}$ by $\frac{cm^3x^3}{a^3b^2}$.

$$\frac{a^2b}{m^2x^3} \times \frac{cm^3x^3}{a^3b^2} = \frac{a^2bcm^3x^3}{a^3b^2m^2x^3} = \frac{a^2bm^2x^3}{a^2bm^2x^3} \times \frac{cm}{ab} = \frac{cm}{ab}, \text{ Ans.}$$

III. Required to find the value of $\frac{a^2 - x^2}{x^4 - y^4} \times \frac{x^2 + y^2}{a + x}$.

$$\frac{a^2 - x^2}{x^4 - y^4} \times \frac{x^2 + y^2}{a + x} = \frac{(a + x)(a - x)(x^2 + y^2)}{(a + x)(x^2 - y^2)(x^2 + y^2)}$$

$$= \frac{a - x}{x^2 - y^2}, \text{ Ans.}$$

IV. Required to find the product of

$$\left(\frac{x^2}{a^2} - \frac{bx}{ay} + \frac{b^2}{y^2}\right) \text{ by } \left(\frac{x^2}{a^2} + \frac{bx}{ay} + \frac{b^2}{y^2}\right).$$

$$\frac{x^2}{a^2} - \frac{bx}{ay} + \frac{b^2}{y^2} = \frac{x^2y^2 - abxy + a^2b^2}{a^2y^2};$$

$$\frac{x^2}{a^2} + \frac{bx}{ay} + \frac{b^2}{y^2} = \frac{x^2y^2 + abxy + a^2b^2}{a^2y^2}.$$

$$\frac{x^2y^2 - abxy + a^2b^2}{a^2y^2} \times \frac{x^2y^2 + abxy + a^2b^2}{a^2y^2}$$

$$= \frac{x^4y^4 + a^2b^2x^2y^2 + a}{a^4y^4}$$

This result may be written

$$\frac{x^4y^4}{a^4y^4} + \frac{a^2b^2x^2y^2}{a^4y^4} + \frac{a^4b^4}{a^4y^4};$$

reduced to simplest form it is

$$\frac{x^4}{a^4} + \frac{b^2x^2}{a^2y^2} + \frac{b^4}{y^4}.$$

Rule.—*Multiply the numerators together for the* merator *of the result. Multiply the denomina* together *for the denominator of the result. Reduce* fraction *thus obtained to its simplest form.*

EXERCISES.

175. ***Find the value of the following:***

1. $\frac{a^2}{b} \times \frac{mn}{c}$.

2. $\frac{5x^2y}{3ab^2c} \times \frac{15abn}{20x^3y}$.

3. $\frac{2a}{3b} \times \frac{6bc}{5a^2}$.

4. $\frac{a^2}{bc} \times \frac{b^2}{ac} \times \frac{c^2}{ab}$.

5. $\frac{6a}{2} \times \frac{5x}{3} \times \frac{8}{5ax}$.

6. $\frac{3(a+x)}{2} \times \frac{4x}{a+x}$.

7. $\frac{a-x}{x^2} \times \frac{a^2}{a^2-x^2}$.

8. $\frac{2a}{3(a-b)} \times \frac{6(a^2-b^2)}{7}$.

9. $\frac{3x^3}{5x-10} \times \frac{30x-60}{4}$.

10. $\frac{3x^2-x}{5} \times \frac{10}{2x^3-4x}$.

11. $\frac{ax}{a+x} \times \left(\frac{x}{a}-\frac{a}{x}\right)$.

12. $\left(b+\frac{a^2}{b}\right) \times \left(a-\frac{b^2}{a}\right)$.

13. $\frac{(a-b)^2}{b+a} \times \frac{b}{x(a-b)}$.

14. $\left(a-\frac{x^2}{a}\right) \times \left(\frac{a}{x}+\frac{x}{a}\right)$.

15. $\frac{a^2-b^2}{a^3-a^2b} \times \frac{a^2b}{a^3+b^3}$.

16. $\frac{x^2+xy}{x^3+y^3} \times \frac{x^3-y^3}{xy(x+y)}$.

17. $\frac{x+1}{x-1} \times \frac{x+2}{x^2-1} \times \frac{x-1}{(x+2)^2}$.

18. $\frac{27+x^3}{y^3-8} \times \frac{y^2+2y+4}{x^2-3x+9}$.

19. $\frac{1-x^2}{1+y} \times \frac{1-y^2}{x+x^2} \times \left(1+\frac{x}{1-x}\right)$.

20. $\frac{3ax}{4by} \times \frac{a^2-x^2}{c^2-x^2} \times \frac{bc+bx}{a^2+ax} \times \frac{c-x}{a-x}$.

21. $\left(b+\frac{bx}{a}\right) \times \left(1-\frac{a}{a+x}\right) \times \frac{a}{bx}$.

22. $\frac{a^3-64}{x^3+216} \times \frac{x^2-6x+36}{a^2+4a+16}$.

23. $\frac{a^2-b^2}{a^2+b^2} \times \frac{a^4-b^4}{(a+b)^2} \times \frac{a+b}{(a-b)^2}$.

24. $\frac{x(a-x)}{a^2+2ax+x^2} \times \frac{a(a+x)}{a^2-2ax+x^2}$.

25. $\dfrac{a^4 - b^4}{a^2 - 2ab + b^2} \times \dfrac{a - b}{a^2 + ab}$.

26. $\dfrac{a^2 - x^2}{m^2 + mn + n^2} \times \dfrac{m^3 - n^3}{a^3 + x^3} \times \dfrac{a^2 - ax + x^2}{m^2 - n^2}$.

27. $\dfrac{x^3 + 2x + 4}{x^3 - 2x + 4} \times \dfrac{x^3 + 8}{x^3 - 8}$.

28. $\left(\dfrac{x^2}{a^2} - \dfrac{x}{a} + 1\right) \times \left(\dfrac{x^2}{a^2} + \dfrac{x}{a} + 1\right)$.

29. $\dfrac{x^2 - 9x + 20}{x^2 - 6x} \times \dfrac{x^2 - 13x + 42}{x^2 - 5x}$.

30. $\left\{1 - \dfrac{a - b}{a + b}\right\} \times \left\{2 + \dfrac{2b}{a - b}\right\}$.

31. $\dfrac{a^2 - x^2}{a + b} \times \dfrac{a^2 - b^2}{ax + x^2} \times \left\{a + \dfrac{ax}{a - x}\right\}$.

32. $\dfrac{x(a - x)}{a^2 + 2ax + x^2} \times \dfrac{x(a + x)}{a^2 - 2ax + x^2}$.

33. $\dfrac{1 - x^2}{1 + y} \times \dfrac{1 - y^2}{x + x^2} \times \left\{1 - \dfrac{x}{x - 1}\right\}$.

34. $\left\{a - \dfrac{a^2 + x^2}{a}\right\} \times \left\{x + \dfrac{a^2 - x^2}{x}\right\}$.

35. $\dfrac{x^2 + 2x - 3}{x^2 + 5x + 6} \times \dfrac{4x^2 - 12x - 40}{3x^2 - 18x + 15}$.

36. $\left\{\dfrac{x^2}{x^2 - y^2} - \dfrac{y^2}{x^2 + y^2}\right\} \times \left\{\dfrac{(x^2 - y^2)^2}{(x^2 - y^2)^2 + (x^2 + y^2)^2}\right\}$.

37. $\dfrac{a^2 - ab + b^2}{a^3 - 3ab(a - b) - b^3} \times \dfrac{a^2 - b^2}{a^3 + b^3}$.

DIVISION.

176. Division of literal fractions is precisely similar to that of numerical fractions.

Thus, $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$; $\frac{a}{b} \div \frac{m}{n} = \frac{a}{b} \times \frac{n}{m} = \frac{an}{bm}$.

Rule.—*Invert the divisor and proceed as in multiplication of fractions.*

EXERCISES.

177. *1.* $\frac{a}{b} \div \frac{c}{d}$.

2. $\frac{a^2x}{b^2y} \div \frac{ax^2}{by^2}$.

3. $\frac{amn}{p^2q^2} \div \frac{m^3n^2}{ap^2q}$.

4. $\frac{2}{x-y} \div \frac{3}{x+y}$.

5. $\frac{a^2b}{a^2-b^2} \div \frac{ab^2}{a^2+ab}$.

6. $\frac{x^3-y^3}{(x^2+y^2)^2} \div \frac{x-y}{x^2+y^2}$.

7. $\frac{2y^2}{x^3+y^3} \div \frac{y}{y+x}$.

8. $\frac{ax-x^2}{(a+x)^2} \div \frac{x^2}{a^2-x^2}$.

9. $\frac{(x^2-y^2)\,xy}{(a+b)^2} \div \frac{xy^2\,(x+y)}{a^2+ab}$.

10. $\frac{x^2-5x+6}{x^2-5x} \div \frac{x^2-3x}{x^2-6x+5}$.

11. $\frac{x^4-y^4}{x^2-2xy+y^2} \div \frac{x^2+xy}{x-y}$.

12. $\frac{2m^2n}{m^2-n^2} \div \frac{3mn^2}{m^2+mn}$.

13. $\frac{x^4-y^4}{a^2b+ab^2} \times \frac{a+b}{(x+y)^2} \div \frac{(x-y)^2}{ab}$.

14. $\left(\frac{x^2}{y^3}+\frac{1}{x}\right) \div \left(\frac{x}{y^2}-\frac{1}{y}+\frac{1}{x}\right)$.

15. $\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right) \div \left(\frac{x+y}{2x-2y}-\frac{x-y}{2x+2y}\right)$.

16. $\frac{x^2-6x+8}{x^2-4x+3} \times \frac{x^2-5x+6}{x^2-2x+8} \div \frac{(x-2)^2}{x-1}$.

178. A complex fraction is a fraction that has a fraction in its numerator, or in its denominator, or in both.

A complex fraction is simply an expression of division of fractions, and may be simplified by treating it as such.

Thus, $\frac{a+\frac{b}{c}}{d+\frac{e}{f}} = \left(a+\frac{b}{c}\right) \div \left(d+\frac{e}{f}\right) = \frac{ac+b}{c} \div \frac{df+e}{f}$

$= \frac{ac+b}{c} \times \frac{f}{df+e} = \frac{f(ac+b)}{c(df+e)}$.

179. ***Simplify the following complex fractions:***

1. $\frac{\frac{4a^2b}{7mn^3}}{\frac{28ab^3}{21m^2n}}$.

2. $\frac{\frac{2m^2n}{m^3-n^2}}{\frac{3mn^2}{m^2+mn}}$.

3. $\frac{\frac{(x^2-y^2)\,xy}{(a+b)^2}}{\frac{(x+y)\,xy^2}{a^2+ab}}$.

4. $\frac{\frac{a^2b}{a^2-b^2}}{\frac{ab^2}{a^2+ab}}$.

5. $\frac{\frac{x^4-y^4}{x^2-2xy+y^2}}{\frac{x^2+xy}{x-y}}$.

6. $\frac{\frac{x^2}{2a^2}+\frac{6a^2}{x^2}-4}{\frac{x}{2a}-\frac{3a}{x}}$.

7. $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}$.

8. $\dfrac{\dfrac{x^2-5x+6}{x^2-5x}}{\dfrac{x^2-3x}{x^2-6x+5}}$.

9. $\dfrac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}}$

10. $1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}$.

11. $\dfrac{1+\dfrac{a-1}{a+1}}{1-\dfrac{a-1}{a+1}}$.

12. $\dfrac{1-\dfrac{2pq}{p^2+q^2}}{\dfrac{p^3-q^3}{p-q}-3pq}$.

13. $\dfrac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

14. $\dfrac{\dfrac{3x}{2}+\dfrac{x-1}{3}}{\dfrac{13}{6}(x+1)-\dfrac{x}{3}-\dfrac{5}{2}}$.

15. $\dfrac{\dfrac{2x}{x+y}+\dfrac{y}{x-y}-\dfrac{y^2}{x^2-y^2}}{\dfrac{1}{x+y}+\dfrac{x}{x^2-y^2}}$.

16. $\dfrac{\dfrac{x}{x-y}-\dfrac{y}{x+y}}{\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{x^2-y^2}}$

17. $\dfrac{\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}}{\dfrac{a^2-b^2}{a^2+b^2}+\dfrac{a^2+b^2}{a^2-b^2}}$.

SIMPLE EQUATIONS.

180. The **degree of an equation** is determined by th highest power of the unknown quantity in any term, or b the greatest number of unknown factors in any term.

Thus, $3x - 5y = 12$ is an equation of the **first degree** $5x^2 - 3xy = 12x + 20$ and $4xy - 5yz = 3x^2 - 20$ are equa tions of the **second degree**; $3x^3 - 5xy^2 = 2y - 5xyz$ is an equation of the **third degree**.

181. A **simple equation** is an equation of the first degree.

182. The **first**, or **left member**, of an equation is the expression on the left side of the sign of equality; the **second**, or **right member**, is the expression on the right side of the sign of equality.

183. A **numerical equation** is an equation in which the known quantities are represented by figures.

Thus, $5x = 30$, $x + y = 7$, $6x^2 + 5xy = 100$, are numerical equations.

184. A **literal equation** is an equation in which some or all of the known quantities are represented by letters.

Thus, $ax + 5 = b$, $x^3 + ax^2 = 24$, $ax + b = cx + d$, are literal equations.

185. An **identical equation** is an equation whose members are alike in form, or may be made so by performing the operations indicated.

Thus, $(5 \times 2) + 8 = 18$, $ax + 5 = ax + 5$, $(a - x)(a + x) = a^2 - x^2$, are identical equations.

SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

186. In the solution of an equation, the operations depend upon the following principles:

I. *If equal quantities be added to both members of an equation, the resulting sums will form an equation.*

Thus, if any number, as 10, be added to each of two equal expressions, it will have the effect of making each expression greater by 10, but they will still be equal.

II. *If equal quantities be subtracted from both members of an equation, the resulting remainders will form an equation.*

Thus, if any number, as 10, be subtracted from each of two equal expressions, it will have the effect of making each expression less by 10, but they will still be equal.

III. *If both members of an equation be multiplied by the same number, the products will form an equation.*

Thus, if equal expressions be multiplied by any number, as 10, the results will be 10 times as great, but still equal.

IV. *If both members of an equation be divided by the same number, the quotients will form an equation.*

Thus, if equal expressions be divided by any number, as 10, the quotients will be only $\frac{1}{10}$ as large, but still equal.

187. It is convenient to have all the unknown terms of an equation in the left member and all the known terms in the right member. This may be effected by Principles I. or II., and is called

TRANSPOSITION.

Given the equation

$$5x - 12 = 3x + 8.$$

Subtracting $3x$ from both members,

$$5x - 3x - 12 = 3x - 3x + 8.$$

Or, $$2x - 12 = 8.$$

Adding 12 to both members,

$$2x - 12 + 12 = 8 + 12.$$

Or, $$2x = 8 + 12 = 20.$$

Hence, *Any term in an equation may be transposed from one member to the other, provided its sign be changed.*

188. An equation may be cleared of fractions by the application of Prin. III.

CLEARING OF FRACTIONS.

Given the equation

$$\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 25.$$

Multiplying both members by 12, the L. C. M. of the denominators 2, 3, and 4,

$$12\left(\frac{x}{2} - \frac{x}{3} + \frac{x}{4}\right) = 12 \times 25.$$

Or, $$6x - 4x + 3x = 300.$$

An equation may be cleared of fractions by multiplying every term by the L. C. M. *of the denominators.*

189. After transposing and uniting terms, the value of the unknown quantity is found by dividing both members by the co-efficient of the unknown quantity, in accordance with Prin. IV.

Thus, in the foregoing equation,

$$6x - 4x + 3x = 300.$$

Uniting terms, $$5x = 300.$$

Dividing by 5, the co-efficient of x,

$$x = 60.$$

190. To solve a numerical simple equation containing one unknown quantity.

I. Required to solve the equation,

$$6x + 2(11 - x) = 3(19 - x).$$

Performing indicated operations,

$$6x + 22 - 2x = 57 - 3x.$$

Adding $3x$ and -22 to both members, or what is same, adding $3x - 22$ to both members,

$$6x + 22 - 2x + (3x - 22) = 57 - 3x + (3x - 22).$$

Or, $$6x - 2x + 3x = 57 - 22;$$

whence, $$7x = 35.$$

Hence, $$x = 5.$$

II. Required the value of x in the equation,

$$\frac{x+3}{2} - \frac{x-2}{3} - \frac{3x-5}{12} = \frac{1}{4}.$$

The equation may be written,

$$\frac{x+3}{2} + \frac{-(x-2)}{3} + \frac{-(3x-5)}{12} = \frac{1}{4};$$

or, $$\frac{x+3}{2} + \frac{-x+2}{3} + \frac{-3x+5}{12} = \frac{1}{4}.$$

Multiplying by 12,

$$6(x+3) + 4(-x+2) + (-3x+5) = 3;$$

or, $$6x + 18 - 4x + 8 - 3x + 5 = 3.$$

Transposing terms,

$$6x - 4x - 3x = 3 - 18 - 8 - 5.$$

Whence, $$-x = -28.$$

Dividing by -1, the co-efficient of x,

$$x = 28.$$

From the foregoing solution it follows:

1. *If an equation contains a fraction preceded by —, in clearing the equation of fractions every term obtained from the numerator of that fraction must have its sign changed.*

2. *The signs of all the terms in both members of an equation may be changed without destroying the equality.*

EXERCISES.

191. ***Find the value of x in the following equations:***

1. $5x + 50 = 4x + 60.$
2. $7x - 6 = 6x - 4.$
3. $8x + 7 = 6x + 27.$
4. $4x - 48 = 24 - 2x.$
5. $27x - 14 = 190 - 41x.$
6. $21x + 7 = 4(x - 3) + 3x + 61.$
7. $2(3x - 1) - 20 + 7x = 5(2x + 1).$
8. $8(2 - x) + 7(1 - 2x) = 37 - 28x.$
9. $3(x - 5) - 5(x - 3) = 21x - 46.$
10. $9(2x - 1) - 7x = 89 - 3(4x + 2).$
11. $7(x - 10) + 54x = 2(1 - x) - 3(5 - 2x).$
12. $\frac{x}{3} + \frac{2x}{5} = 3 - \frac{x}{6}.$

13. $\frac{6x}{7} - \frac{5x}{4} + 31 + \frac{3x}{2} = 0.$

14. $19x - 4x = 17\frac{1}{2} - \frac{7x-2}{2}.$

15. $x + \frac{3x-5}{2} = 12 - \frac{2x-4}{3}.$

16. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} - 17 = x.$

17. $\frac{x+1}{3} - \frac{x+4}{5} = 16 - \frac{x+3}{4}.$

18. $3x - \frac{11x-37}{2} = 5 - \frac{2x+6}{5}.$

19. $\frac{x-1}{7} = 7 - \frac{23-x}{5} - \frac{4+x}{4}.$

20. $7x + 13\frac{3}{4} - \frac{x}{2} = \frac{4x}{5} - 8\frac{3}{4} + \frac{41x}{5}.$

21. $\frac{x-3}{4} - \frac{2x-5}{6} - \frac{41}{60} = \frac{3x-8}{5} - \frac{5x+6}{15}.$

22. $\frac{7x+9}{4} - \left(x - \frac{2x-1}{9}\right) = 7.$

23. $\frac{7+9x}{4} - \left(1 - \frac{2-x}{9}\right) = 7x.$

24. $\frac{3x-11}{4} - \frac{28-9x}{8} = 4x - 14\frac{3}{4}.$

25. $\frac{2(x-8)}{3} - \frac{3(9-x)}{4} - \frac{5(x-11)}{6} = 7 - \frac{3(x-17}{8}$

26. $\dfrac{3x-1}{2x-1}-\dfrac{4x-2}{3x-2}=\dfrac{1}{6}$.

27. $\dfrac{3+x}{3-x}-\dfrac{2+x}{2-x}-\dfrac{1+x}{1-x}=1$.

28. $\dfrac{2}{2x-3}=\dfrac{6}{3x+2}-\dfrac{1}{x-2}$.

29. $5x-[8x-3\{16-6x-(4-5x)\}]=6$.

30. $(x-5)(x-2)-(x-5)(2x-5)+(x+7)(x-2)=0$.

192. It is sometimes best to clear an equation of fractions only partially, and to complete the operation after reducing.

III. Required the value of x in

$$\frac{7(x-1)}{39}+\frac{5(x+6)}{3x+5}=\frac{2\frac{1}{3}x-21\frac{5}{6}}{13}+3\tfrac{1}{2}.$$

Multiplying by 39,

$$7x-7+\frac{195(x+6)}{3x+5}=7x-65\tfrac{1}{2}+136\tfrac{1}{2}.$$

Omitting $7x$ and reducing,

$$\frac{195(x+6)}{3x+5}=78.$$

Dividing by 39, $\dfrac{5(x+6)}{3x+5}=2.$

Clearing of fractions, $5x+30=6x+10.$

Whence, $x=20.$

EXERCISES.

193. ***Find the value of the unknown quantity in t following:***

1. $\frac{9x+20}{36} = \frac{4x-12}{5x-4} + \frac{x}{4}$.

2. $\frac{7x+16}{21} - \frac{x+8}{4x-11} = \frac{x}{3}$.

3. $\frac{6x+7}{9} + \frac{7x-13}{6x+3} = \frac{2x+4}{3}$.

4. $\frac{4x+3}{9} = \frac{8x+19}{18} - \frac{7x-29}{5x-12}$.

5. $\frac{3x-2}{2x-5} - \frac{21-3x}{5} = \frac{6x+13}{10}$.

6. $\frac{6x+31}{15} - \frac{3x+5}{5x-25} = \frac{2x}{5}$.

7. $\frac{4}{x+2} + \frac{7}{x+3} = \frac{37}{x^2+5x+6}$.

8. $\frac{25-\frac{x}{3}}{x+1} + \frac{16x+4\frac{1}{2}}{3x+2} = 5 + \frac{23}{x+1}$.

9. $\frac{1}{3} - \frac{7x-1}{6\frac{1}{2}-3x} = \frac{8\,(x-\frac{1}{2})}{3\,(x-2)}$.

10. $\frac{9x+5}{14} + \frac{8x-7}{6x+2} - \frac{36x+15}{56} = \frac{10\frac{1}{2}}{14}$.

11. $\frac{x-7}{x+7} - \frac{2x-15}{2x-6} + \frac{1}{2\,(x+7)} = 0$.

12. $\dfrac{6x-7\frac{1}{3}}{13-12x}+2x+\dfrac{1+16x}{24}=4\frac{5}{12}-\dfrac{12\frac{2}{3}-8x}{3}.$

13. $\dfrac{18x-19}{28}+\dfrac{11x+21}{6x+14}=\dfrac{9x+15}{14}.$

14. $\dfrac{2x+8\frac{1}{2}}{9}-\dfrac{13x-2}{17x-32}+\dfrac{x}{3}=\dfrac{7x}{12}-\dfrac{x+16}{36}.$

15. $\dfrac{10x+17}{18}-\dfrac{12x+2}{13x-16}=\dfrac{5x-4}{9}.$

16. $\dfrac{5}{1-5x}+\dfrac{4}{2x-1}=\dfrac{3}{3x-1}.$

17. $\dfrac{6-5x}{15}-\dfrac{7-2x^2}{14\,(x-1)}=\dfrac{1+3x}{21}-\dfrac{2x-2\frac{1}{2}}{6}+\dfrac{1}{105}.$

18. $\dfrac{1}{x-1}-\dfrac{1}{2\,(x+1)}-\dfrac{x+3}{2\,(x^2+1)}=\dfrac{6}{x^4-1}.$

194. To solve a literal simple equation containing one unknown quantity.

I. Required the value of x in

$$\frac{x^2-a^2}{bx}-\frac{a-x}{b}=\frac{2x}{b}-\frac{a}{x}.$$

Multiplying by bx, the L. C. D. of the denominators,

$$x^2-a^2-ax+x^2=2x^2-ab.$$

Subtracting $2x^2$ from each member,

$$-a^2-ax=-ab.$$

Transposing $-a^2$ and dividing by $-a$,

$$x=b-a.$$

II. Required the value of x in

$$(a + x)(b + x) - a(b + c) = \frac{a^2c}{b} + x^2.$$

Performing the multiplication indicated,

$$ab + ax + bx + x^2 - ab - ac = \frac{a^2c}{b} + x^2.$$

Subtracting x^2 from each side, and canceling $+ab$ and $-a$

$$ax + bx - ac = \frac{a^2c}{b}.$$

Clearing of fractions,

$$abx + b^2x - abc = a^2c.$$

Transposing $-abc$, and factoring,

$$(a + b)bx = ac(a + b).$$

Dividing by $a + b$, $\quad bx = ac.$

Whence, dividing by b, $x = \frac{ac}{b}$.

EXERCISES.

195. ***Find the value of the unknown quantity in t following:***

1. $x - a = (b - a)x.$
2. $\frac{x}{a} - \frac{x - b}{c} = \frac{de}{ac}.$
3. $\frac{x}{a - b} - \frac{x}{a + b} =$
4. $\frac{x}{a} + c = \frac{x}{b} - d.$
5. $ax - c = d - bx.$
6. $3ax - 4ab = 2ax - 6ac.$

7. $ax^2 + abx = cdx.$

8. $ax^2 + bx = dx^2 + cx.$

9. $\frac{a}{bx} + \frac{b}{ax} = a^2 + b^2.$

10. $\frac{a(a-x)}{b} - \frac{b(b+x)}{a} = x.$

11. $\frac{a(x-a)}{b} + \frac{b(x-b)}{a} = x.$

12. $a(x+a) - b(x-b) = 3ax + (a-b)^2.$

13. $\frac{a}{x} + \frac{b}{a} = \frac{1}{3x}.$

14. $x + \frac{ax}{b} + \frac{cx}{b} = m.$

15. $x(x-a) + x(x-b) = 2(x-a)(x-b).$

16. $(x-a)(x-b) = (x-a-b)^2.$

17. $\frac{1}{a(b-x)} + \frac{1}{b(c-x)} = \frac{1}{a(c-x)}.$

18. $\frac{x-a}{a-b} - \frac{x+a}{a+b} = \frac{2ax}{a^2-b^2}.$

19. $(a+x)(b+x) - a(b+c) = \frac{a^2c}{b} + x^2.$

20. $\frac{2x+a}{b} - \frac{x-b}{a} = \frac{3ax + (a-b)^2}{ab}.$

21. $\frac{a^2+x^2}{ax} + \frac{a(x-a)}{x(x+a)} + \frac{ax}{x^2-a^2} + 2 = \frac{x(x+a)}{a(x-a)}.$

22. $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0.$

23. $(a-b)(x-c)-(b-c)(x-a)=(c-a)(x-$

MISCELLANEOUS EXAMPLES.

196. ***Find the value of x in the following equatio***

1. $\frac{x+7}{3}+\frac{2x+6}{14}=\frac{6x-10}{7}.$

2. $\frac{x}{7}+\frac{x}{3}=\frac{13x+14}{28}.$

3. $\frac{3x+6}{13}=\frac{7x+13}{30}.$

4. $2+\frac{x-7}{3}=\frac{3x+3}{10}.$

5. $7-\left(\frac{x-7}{12}+\frac{7x-1}{20}\right)=\frac{5x+7}{18}-2.$

6. $\frac{9-x}{3}+\frac{3x+5}{10}=\frac{7x-5}{25}-1.$

7. $\frac{11(x-1)}{4}+\frac{11-x}{3}=\frac{9(x+3)}{12}+\frac{7(x+1)}{6}.$

8. $\frac{1}{4}(x-7)+\frac{2}{5}(x+3)=\frac{2}{7}(2x+9)-\frac{1}{12}(x-$

9. $5x+\frac{3x-1}{7}=\frac{5}{9}(6x+15)+\frac{1}{7}(3x-1).$

10. $\frac{7-x}{5}+\frac{3x-5}{7}+\frac{7x-3}{8}+\frac{14-4x}{13}=-1.$

11. $\frac{x-7}{10} - \frac{3x-1}{35} - \frac{4x+2}{45} = 1.$

12. $\frac{(3x+5)(5x+3)}{23} - \left(3x + \frac{x+3}{7}\right) = \frac{10x(3x-2)}{46} - 6.$

13. $\frac{1}{2}(1+25x) - \left(1 - \frac{4-2x}{9}\right) = 12x.$

14. $\frac{1}{4}(9x-1) - \left(2x - \frac{3x-7}{2}\right) = 5.$

15. $\frac{1}{x-4} - \frac{1}{x-3} = \frac{1}{x-2} - \frac{1}{x-1}.$

16. $\frac{x-3}{2} - \frac{x}{2} + 16\frac{2}{3} = \frac{x}{9} + \frac{4x-171}{29}.$

17. $(2-x)^2 + 8 = [2(1+x) - x]x.$

18. $\frac{5x+3}{3} - \left(12 + \frac{4x+9}{7}\right) + 13 = 4.$

19. $5.6x - \frac{.48x - 3.7}{.5} - .8x = 4x + 5.8.$

20. $\frac{2}{ac} + \frac{2}{ab} + \frac{4}{bc} = 2ax + bx + cx.$

21. $\frac{b(c^2+x^2)}{cx} = bd + \frac{bx}{c}.$

22. $\frac{7x+10}{2x-10} - \frac{4x-29}{57} = \frac{x+45}{19} - \frac{7x-122}{57}.$

23. $\frac{x-3}{x-5} - \frac{x+6}{x+4} = \frac{5x-2}{x^2-x-20}.$

24. $(2x+3)(3x+2)+(5x+7)(3-x)$
$=(7x+27)(x-7)-2(3x-7)(x-$

25. $7x-\left(10x-\frac{9x+5}{2x}\right)=5-3x.$

26. $\frac{a-x}{x-b}+\frac{x+c}{b+x}=\frac{a+x}{b-x}+\frac{x-c}{b+x}.$

27. $(x+3)^2-\frac{8x(x+5)}{10}=7x-\left(3x-\frac{x(x-5\frac{1}{2})}{5}\right.$

28. $\frac{1}{amn+mnx}+\frac{1}{abc+bcx}=\frac{1}{a^2mc-mcx^2}.$

29. $\frac{a-bx}{b}-\frac{b-ax}{a}=ax+bx.$

30. $\frac{ax}{b}-\frac{bx}{a}=a+b.$

31. $(2a+b)(x+1)=(2b+a)(x-1).$

32. $(a-b+c)(x-a)=(b-a+c)(x+a).$

33. $x+\frac{x-4}{17}=3x-\frac{5-x}{34}-6.$

34. $\frac{4x-11}{8}+17x=\frac{7}{8}\left(19x-\frac{x-13}{7}\right).$

35. $(a-x)(x-b)+(a+x)(x-b)=(a-b)^2.$

36. $(x-7)(2-x)+(1\frac{1}{2}x-1)(\frac{2}{3}x+5)=5\left(\frac{x}{6}+\right.$

37. $(2x-3)^2=\{11-(11\frac{1}{4}-x)\}4x.$

38. $\frac{(5x-7)(1-x)}{3x+2}+\frac{2\frac{1}{2}x+4}{1\frac{1}{3}}=20.$

PROBLEMS LEADING TO EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

197. In the application of the equation to the solution of problems two things are necessary:

1. *The statement of the conditions of the problems in the form of equations.*
2. *The solution of the equations thus obtained.*

DIRECTIONS FOR STATING PROBLEMS.

1. Study the problem so as thoroughly to understand its meaning.
2. Represent the unknown quantities by final letters of the alphabet.
3. Employing these final letters as if they were known values, express the relations of the known and the unknown quantities in the form of an equation.

PROBLEMS.

198. I. What number is 165 greater than the sum of its third and its fourth parts?

Solution.

Let x = the number.

Then, $$x - \left(\frac{x}{3} + \frac{x}{4}\right) = 165.$$

Clearing of fractions,

$$12x - 4x - 3x = 1980.$$

Whence, $$5x = 1980,$$

and $$x = 396, \text{ the number.}$$

1. A man spent $\frac{3}{10}$ of his money for a suit of clothes an of it for a watch, and had $54 left. How much money h he at first?

2. What number diminished by the sum of its half and third leaves a remainder of 36?

3. A farmer put $\frac{1}{3}$ of his land in corn, $\frac{1}{4}$ of it in whe and $\frac{1}{8}$ of it in oats. The remainder, amounting to $37\frac{1}{2}$ ac was meadow. How large was the farm?

4. I expended $\frac{2}{5}$ of my annual salary for rent, $\frac{1}{4}$ of it food, $\frac{1}{12}$ of it for taxes, $\frac{1}{8}$ of it for clothing, and had $500 l What was my salary?

5. A hat cost $\frac{2}{5}$ as much as a coat, and together they c $21. What did each cost?

II. The sum of two numbers is 533 and their difference 167. What are the numbers?

Solution.

Let $x =$ the greater number.

Then $x - 167 =$ the less number.

And $x + (x - 167) = 533.$

Transposing and uniting terms,

$2x = 700.$

Whence, $x = 350$, the greater number.

And $x - 167 = 183$, the smaller number.

6. A house and lot cost $3775, and the house cost $14 more than the lot. What did each cost?

7. John and his father earned $165 during a certa month, and John earned $37.60 less than his father. H much did each earn?

8. A gentleman paid \$600 for a horse, carriage, and sleigh. The horse cost \$25 more than the carriage and \$65 more than the sleigh. What was the cost of each ?

9. A man dying left a fortune of \$15800 to his wife, son, and daughter. The son was to receive \$2500 more than the daughter, and \$3300 less than the wife. What was the share of each ?

10. A man drove 155 miles in three days. On the second day he drove 15 miles farther than on the first, and on the third 20 miles farther than on the second. How far did he drive each day ?

III. A man worked 25 days during a certain month. For some of the time he received \$1.45 a day, and for the rest of the time \$1.65 a day. How many days did he work for \$1.45 a day, if during the month he earned \$38.65 ?

Solution.

Let x = number of days at \$1.45.

Then $25 - x =$ " " \$1.65.

So that, $\$1.45x + \$1.65\,(25 - x) = \$38.65.$

Multiplying by 100 and performing indicated operation,

$$145x + 4125 - 165x = 3865.$$

Whence, $$-20x = -260.$$

And $$x = 13 \text{ days.}$$

11. A grocer mixed tea worth \$.40 a pound with tea worth \$.50 a pound, so that 100 pounds of the mixture was worth \$44.50. How many pounds of each did he take ?

12. From two towns 204 miles apart, two travelers set o at the same time, with the intention of meeting. One of the goes 15 miles and the other 19 miles a day. In how ma days will they meet ?

13. A foreman, 5 journeymen, and 3 apprentices earn $333.60 in a certain time. The foreman got $5 a day, journeyman $3.75, and an apprentice $1.35. How long d they work ?

14. In a certain family there were four children born intervals of three years. At the end of each year of th lives their father deposited $5 in a bank for each child. H old was each child at the time of the last deposit if they h in the bank $250 in all ?

IV. A certain sum of money was divided among A, B, a C. A and B received $65; A and C, $92; and B and $87. How much did each receive ?

Solution.

Let x = what A received.

Then $65 - x =$ " B "

And $92 - x =$ " C "

But B and C received $87; so that

$$(65 - x) + (92 - x) = 87.$$

Whence, $$-2x = -70.$$

And $$x = 35,$$

$$65 - x = 30,$$

$$92 - x = 57.$$

Hence, $35 = A's share; $30 = B's share; $57 = C share.

15. A man bought an overcoat, a suit of clothes, and a pair of boots. The overcoat and the suit of clothes cost together \$70; the overcoat and boots, \$39; and the suit of clothes and boots, \$51. How much did each cost?

16. A man and his oldest son can together earn \$54 a week; the man and his youngest son, \$48 a week; and the two sons, \$32 a week. How much can each alone earn per week?

17. A bushel each of wheat and rye cost \$2.05; of wheat and oats, \$1.60; of rye and oats, \$1.39. Find the cost of a bushel of each.

V. A is three times as old as B, and 5 years ago the sum of their ages was equal to A's present age. How old is each?

Solution.

Let x = B's present age.

Then $3x$ = A's " "

$$(x - 5) + (3x - 5) = 3x.$$

Whence, $4x - 3x = 10,$

or $x = 10,$

and $3x = 30.$

Hence, A's present age is 10 years, and B's 30 years.

18. A is twice as old as B, but 15 years ago the sum of their ages was 84 years. How old is each?

19. Mr. Brown's age is $1\frac{1}{3}$ times his wife's age, but 20 years ago, when they were married, his age was $1\frac{3}{4}$ times his wife's age. What is the age of each?

VI. Two numbers are to each other as 2 to 5; if 14 be added to each, the sums are to each other as 5 to 9 respectively. What are the numbers?

Solution.

Let $2x$ = the first number.

Then $5x$ = the second number.

$$2x + 14 : 5x + 14 :: 5 : 9.$$

Since the product of the means equals the product of the extremes,

$$18x + 126 = 25x + 70.$$

Whence, $7x = 56$, and $x = 8$.

$2x = 16$, the first number,

$5x = 40$, the second number.

20. What two numbers that are to each other as 3 to 4 will, if each is increased by 50, give sums that are to each other as 5 to 6 respectively?

21. A man bought a house and a lot, the prices of which were as 5 to 3 respectively. If each had cost $500 less, their prices would have been as 9 to 5. What did he pay for each?

22. A and B each owns half of a flock of sheep. In dividing, A gets 72 sheep and B 94. In order to make the division equal, B must pay A $44. What is the value of a sheep?

23. Divide $400 among three persons so that the second shall have twice as much as the first, and the third $25 less than the second.

24. Two persons, A and B, invest equal sums in business. A gains $875, and B loses $1575. A's money is now three times as much as B's. How much did each invest?

25. A man bought some sheep for $480. After 12 of them died, he sold $\frac{1}{4}$ of the remainder at cost for $108. How many sheep did he buy?

26. A gentleman wished to distribute some money among some children. He lacked $.25 of having enough to give each $.25. If he should give each $.20 he would have $.80 left. How many children were there?

27. A man bought a watch and chain for $185. The cost of the chain lacked $18 of being equal to $\frac{1}{5}$ of the cost of the watch. What was the cost of each?

28. Seven times the value of A's house is equal to 9 times the value of his lot. What is the value of each if together they are worth $12800?

29. A steamer and its cargo are together worth $127400; what is the value of each, if the steamer lacks $9100 of being worth $2\frac{1}{4}$ times as much as the cargo?

30. I had $5000. I loaned some of it at 5%, and the remainder at 6%. The annual interest received was $280. How much drew 5%?

31. A composition of copper and tin containing 100 cubic inches weighed 505 ounces. How many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and a cubic inch of tin to weigh $4\frac{1}{4}$ ounces?

32. A silversmith has three pieces of metal, which together weigh 48 ounces. The second weighs 12 ounces more than the first, and the third 9 ounces more than the second. What are their respective weights?

33. What number is that the quadruple of which, diminished by 26, exceeds 150 by as much as 7 times the number diminished by 20 is less than 200?

34. Divide $155 among A, B, and C, so that B shall have $\$5\frac{1}{2}$ more than A, and C $\frac{2}{3}$ as much as A and B together.

35. A window 8 feet in length is so situated in a house that from the lower edge of the window to the top of the house is $\frac{3}{7}$ of the entire height of the house; and from the top of the window to the ground is $\frac{3}{4}\frac{1}{2}$ of the height. What is the height of the house?

36. A sells B 17 sheep, and then has only half as many as B; but at first he had 11 more than B. How many had each at first?

37. Divide 96 cents into two parts, so that the first may be as many times five cents as the second is times three cents.

38. A, B, and C agree to do a piece of work for $100. A works $2\frac{1}{2}$ times as long as B, and B $\frac{4}{5}$ as long as C. How much money should each receive?

39. John earns 35 cents per day less than his father, and in 11 days the father earns $12.10 more than the son earns in 6 days. Find the daily wages of each.

40. A's fortune is three times B's, but after each had lost half, A gave B $250, when they had equal sums. How much had they at first?

41. The sum of two numbers is 97 and their difference 17. Required the numbers.

42. A gentleman divided a certain sum of money among four persons. He gave the first 10 dollars; the second, one-half of the whole; the third, one-fourth of the whole; the fourth, 5 dollars. Required the whole sum.

43. Said A to B, "My age is 10 years more than yours, but in 10 years mine will be twice what yours was 5 years ago." Required their ages.

44. A gentleman invests $\frac{1}{3}$ of his money at 8%, $\frac{1}{4}$ at $4\frac{1}{2}$%, $\frac{1}{6}$ at $7\frac{1}{2}$%, and the remainder at 5%, and realizes an annual income of $680. How much money has he?

45. How many 10-cent pieces and how many 25-cent pieces must be taken so that 70 pieces shall make \$10 ?

46. The sum of three numbers is 272. The first is $1\frac{1}{2}$ times the second, and the third is 17 more than $\frac{4}{5}$ of the sum of the other two. Required the numbers.

47. The owners of a steamboat make \$30 per day each, but if there were fewer owners by 5, each would make \$10 per day more. How many owners are there ?

48. A man bought a horse, carriage, and harness for \$290. For the horse he paid \$10 more than the cost of the other two, and the harness cost \$10 less than half as much as the carriage. Find the cost of each.

49. A gentleman has a yearly income such that $\frac{3}{4}$ of it increased by \$200 is equal to his yearly expenditure. In 5 years he can save a sum equal to \$300 more than $\frac{1}{10}$ of his yearly income. How much does he earn, and how much does he spend per annum ?

50. A man agreed to work for 30 days, on condition that for each day he worked he was to receive \$5, and for each day he was idle he was to pay \$3.75 to his employer. At the end of the time he received \$10. Find the number of working and idle days.

51. Two boys had equal sums of money. One bought some lemons at 5 cents each and had 4 cents left ; the other bought $1\frac{2}{3}$ times as many oranges at 2 cents each and had 29 cents left. How much money had each ?

52. A man has a certain sum of money to invest. It is found that the annual interest at 7% on \$500 more than $\frac{4}{5}$ of it exceeds by \$49 that at 6% on \$700 less than $\frac{1}{3}$ of it. Required the sum.

53. A merchant has a mixture consisting of 100 pounds of 15 and 11 cent sugar, which he can afford to sell at $12\frac{1}{2}$

cents per pound. How many pounds of each kind are there?

54. A company of 90 persons consists of men, women, and children. The men are 4 more in number than the women, the children 10 more than the men and women together. How many were there of each?

55. A father gives to his five sons $1000, which they are to divide according to their ages, so that each elder son shall receive $20 more than his next younger brother. What is the share of the youngest?

56. A's age when he married was to his wife's age as $\frac{3}{4}$ to $\frac{4}{5}$, but after 12 years their ages are as 13 to 12. How old were they when married?

57. A and B set out at the same time to meet each other, A traveling $5\frac{1}{2}$ miles and B $3\frac{1}{2}$ miles per hour. When they meet, A has passed the half-way point 66 miles. How far apart were they at starting?

58. In a company of soldiers, there is one officer to every 15 soldiers, and if the whole number of men be divided by the number of officers less 10, the quotient will be 20. What is the number of men in the company?

59. Divide $5000 into two such parts that the first on interest for 6 years at 10% may amount to the same sum as the second in 8 years at 5%.

60. A man bought a house and lot for $5000, and paid $4000 in cash. Six months afterward he paid the remainder in equal quantities of potatoes and apples at the rate of 75 and 50 cents respectively. Allowing interest at six per cent., how many bushels did he give?

61. A person distributes $1.90 among 16 beggars, giving to some 13 cents each and to the rest 11 cents each. How many received 13 cents each?

62. A puts $2000 at interest in two different sums, the smaller at 7 per cent., and the greater at 6 per cent. The interest on the former for 9 years is the same as that on the latter for 7. What is each sum?

63. A farmer having hired a laborer, on condition that for every day he worked he should receive 50 cents, and for every day he was idle he should forfeit 20 cents, finds, at the end of 420 days, that neither is indebted to the other. How many days did he labor?

64. A person wishes to pay $1.51 with 39 pieces of money, some 3-cent and the rest 5-cent pieces; how many of each must he take?

65. A tree 100 feet high is broken into three pieces by falling. Twice the upper piece lacks 4 feet of being equal to 3 times the lower piece, and twice the lower piece increased by 3 feet is equal to the middle piece. Required the length of the pieces.

66. How far down a river whose current is 3 miles per hour can a man go whose rate of rowing in still water is 7 miles per hour, so that he may be back in 7 hours?

67. The number of acres a farmer planted with corn is to the number he planted with potatoes as 4 to 3; but if he had planted 6 acres less of corn, and $\frac{1}{2}$ as many acres of potatoes $+ 15\frac{1}{2}$ acres, the ratio would have been as 3 to 10. How many acres did he plant of each?

68. I paid a bill of $5.75 with 16 pieces of money. Some of them were 50-cent pieces and the rest were 25-cent pieces. How many of each were there?

69. A man bought a farm for $75 an acre. Reserving 10 acres, he sold the remainder at $80 an acre and received for it $40 more than the farm cost him. How many acres were there in the farm?

SIMULTANEOUS SIMPLE EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

199. Independent equations are such as cannot be made to assume the same form by any proper reduction.

Thus, $5x + 3y = 17$ and $3x + y = 11$ are independent equations. But $5x + 3y = 17$ and $1\frac{1}{3}x + y = 5\frac{2}{3}$ are *dependent*, for the first equals three times the second.

200. Simultaneous equations are such as can be satisfied by the same values of the unknown quantities.

201. An equation is said to be **satisfied** when a substitution of known values for the unknown quantities in it will make the equation identical.

Thus, in the equation $7x - 3 = 5x + 11$, if we substitute 7, the value of x, the equation becomes $(7 \times 7) - 3 = (5 \times 7) + 11$, or $46 = 46$.

202. Elimination, in algebra, is the process of obtaining from a given set of equations containing two or more unknown quantities, a new set from which one of the unknown quantities has disappeared.

The quantity that disappears is said to be **eliminated.**

203. This process enables us to obtain from two equations, each containing two unknown quantities, a single equation containing but *one.* The equation thus obtained can be solved as shown in the preceding chapter.

There are several methods of elimination. Those most commonly used are,

I. **Comparison.**
II. **Substitution.**
III. **Addition** or **subtraction.**

I. ELIMINATION BY COMPARISON.

204. Let it be required to find the values of x and y in the equations,

$$5x + 3y = 46, \tag{1}$$

$$2x + 7y = 59. \tag{2}$$

Transposing in both equations the terms containing y,

$$5x = 46 - 3y, \tag{3}$$

$$2x = 59 - 7y. \tag{4}$$

Dividing each equation by the co-efficient of x,

$$x = \frac{46 - 3y}{5}, \tag{5}$$

$$x = \frac{59 - 7y}{2}. \tag{6}$$

Now, since the second members of (5) and (6) are each equal to the same quantity, x, they must themselves be equal. That is,

$$\frac{46 - 3y}{5} = \frac{59 - 7y}{2}. \tag{7}$$

Clearing of fractions,

$$92 - 6y = 295 - 35y.$$

Whence, $29y = 203,$

and $y = 7.$

Substituting the value of y in (1),

$$5x + 21 = 46.$$

Whence, $5x = 25,$

and $x = 5.$

Rule.—I. *Find the values of the same unknown quantity in the two equations, in terms of the other unknown quantity and the known quantities.*

II. *Make the values thus found equal to each other and solve the resulting equation.*

205. By dividing (3) by (4) in the foregoing operation the work is shortened. Thus,

$$\frac{5}{2} = \frac{46 - 3y}{59 - 7y}.$$

Clearing of fractions,

$$92 - 6y = 295 - 35y.$$

The writing of equations (5), (6), and (7) is thus avoided.

EXERCISES.

206. *Solve by comparison:*

1. $2x + 3y = 23;\quad 5x - 2y = 10.$
2. $6x - 2y = 30;\quad 5x - 6y = 12.$
3. $11x + 3y = 100;\quad 4x - 7y = 4.$
4. $5x - 3y = 75;\quad x + 5y = 155.$

5. $x + 9y = 63$; $9x + y = 87$.

6. $x + y = 9$; $3x + 5y = 35$.

7. $2x + 3y = 18$; $3x - 2y = 1$.

8. $x + 2y = 33$; $3x - 5y = 0$.

9. $2x - \frac{y-3}{5} = 4$; $3y + \frac{x-2}{3} = 9$.

10. $x - \frac{y-2}{7} = 5$; $4y - \frac{x+10}{3} = 3$.

11. $\frac{x}{3} + 3y = 7$; $\frac{4x-2}{5} = 3y - 4$.

12. $6x - 5y = 1$; $7x - 4y = 8\frac{1}{2}$.

13. $2x + \frac{y-2}{5} = 21$; $4y + \frac{x-4}{6} = 29$.

14. $\frac{3x}{19} + 5y = 13$; $2x + \frac{4-7y}{2} = 33$.

15. $\frac{x}{7} + \frac{y}{14} = 10\frac{1}{2}$; $2x - y = 7$.

16. $\frac{x+y}{3} - \frac{x-y}{2} = 9$; $\frac{x}{2} + \frac{x+y}{9} = 5$.

17. $\frac{3x}{4} - \frac{2y}{3} = 1$; $\frac{7x}{3} + \frac{5y}{6} = 6$.

18. $\frac{x+y}{3} + x = 15$; $\frac{x-y}{5} + y = 6$.

19. $\frac{7x}{6} + \frac{5y}{3} = 34$; $\frac{7x}{8} + \frac{3y}{4} = \frac{5y}{8} + 12$.

20. $\frac{x+y}{8} - \frac{y-x}{6} = 5$; $\frac{x+y}{4} - \frac{x-y}{3} = 10$.

21. $\frac{7x}{4} + \frac{5y}{8} = 20$; $\frac{3x}{5} + \frac{7y}{4} = 2x - 7$.

22. $\frac{2x + 3y}{5} = 10 - \frac{y}{3}$; $\frac{4y - 3x}{6} = \frac{3x}{4} + 1$.

23. $\frac{1 - 3x}{7} - \frac{1 - 3y}{5} = 2$; $\frac{3x + y}{11} + y = 9$.

24. $2(2x + 3y) = 3(2x - 3y) + 10$;

$4x - 3y = 4(6y - 2x) + 3$.

25. $\frac{9}{x} - \frac{4}{y} = 1$; $\frac{18}{x} + \frac{20}{y} = 16$.

26. $x - 4y = 7$; $\frac{x}{3y} + \frac{11}{10} = \frac{4x - 5y}{5y}$.

II. ELIMINATION BY SUBSTITUTION.

207. Let it be required to find the values of x and y in the equations,

$$x + 7y = 17, \tag{1}$$

$$2x + 5y = 16. \tag{2}$$

From (1), $$x = 17 - 7y, \tag{3}$$

Substituting this value of x in (2),

$$2(17 - 7y) + 5y = 16.$$

Whence, $$-9y = -18,$$

and $$y = 2.$$

Substituting 2 for y in (1),

$$x + 14 = 17.$$

Hence, $$x = 3.$$

Rule.—I. *Find an expression for either of the unknown quantities in one of the equations, and substitute it for that quantity in the other equation.*

II. *Solve the resulting equation, and substitute in one of the given equations the value thus found.*

EXERCISES.

208. ***Solve by substitution:***

1. $3x - 4y = 2$; $7x - 9y = 7$.
2. $5x + 2y = 29$; $6x - 2y = 26$.
3. $3x + 2y = 0$; $3x - 4y = 18$.
4. $5x - 2y = -4$; $4x + y = 28$.
5. $2x - 3y = 0$; $8x - y = 22$.
6. $6x - 5y = 1$; $14x - 8y = 17$.
7. $4x + 10y = 5$; $100x + 40y = 41$.
8. $\frac{3x}{19} + 5y = 13$; $2x - \frac{7y - 4}{2} = 33$.
9. $\frac{x + y}{3} - \frac{x - y}{2} = 9$; $\frac{x}{2} + \frac{x + y}{9} = 5$.
10. $4x + 5y = 40\,(x - y)$; $2x + 5y = 1\frac{1}{2}$.
11. $\frac{x}{3} - \frac{y}{5} = 5$; $\frac{x}{5} + y = 31$.
12. $\frac{x}{7} - \frac{7y}{10} = -20$; $\frac{x}{4} + 3y = 134$.
13. $\frac{5x - 6y}{13} + 3x = 4y - 2$;

$\frac{5x + 6y}{6} - \frac{3x - 2y}{4} = 2y - 2$.

14. $2x - \frac{x-3}{2} = 6$; $5x + 4y - \frac{7y+5x}{2} = 10$.

15. $\frac{x}{15} - \frac{y}{20} = 1$; $\frac{y}{10} - \frac{x}{30} = 1$.

16. $\frac{x}{2} + \frac{y}{3} = 7$; $\frac{x}{3} + \frac{y}{2} = 8$.

17. $\frac{4x+3y}{6} = 8$; $\frac{7y-3x}{2} - y = 11$.

18. $\frac{x}{6} + 6y = 150$; $6x + \frac{y}{6} = 220$.

19. $\frac{74}{x} - \frac{74}{y} = 35$; $\frac{19}{x} + \frac{37}{y} = 10\frac{1}{2}$.

20. $\frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4}$;

$\frac{2x+4}{3} - \frac{2x+y}{8} = \frac{x+5}{4}$.

21. $\frac{7+x}{5} - \frac{2x-y}{4} = 3y - 5$;

$\frac{5y-7}{2} - \frac{3-4x}{6} = 18 - 5x$.

22. $\frac{x+y}{2} - \frac{x-y}{3} = 8$; $\frac{x+y}{3} - \frac{y-x}{4} = 11$.

23. $\frac{x+y}{3} - \frac{y-x}{4} = 59$; $\frac{x}{11} = \frac{3y}{5}$.

24. $\frac{x}{2} - \frac{y}{3} - 1 = 0$; $\frac{2x-1}{2} - \frac{3y-1}{3} = \frac{5}{6}$.

25. $\frac{3x-5y}{3} - \frac{2x-8y-9}{12} = \frac{y}{2} + \frac{7}{12}$;

$3\frac{1}{4}\left(\frac{x}{7} + \frac{y}{4} + 1\frac{1}{3}\right) - 3\frac{1}{3}\left(4x - \frac{y}{8} - 24\right) = 0$.

III. ELIMINATION BY ADDITION OR SUBTRACTION.

209. Let it be required to find the values of x and y in the equations,

$$5x - 6y = -55, \quad (1)$$

$$3x + 4y = 195. \quad (2)$$

Multiplying (1) by 2 and (2) by 3,

$$10x - 12y = -110, \quad (3)$$

$$9x + 12y = 585. \quad (4)$$

Adding (3) and (4), $19x = 475, \quad (5)$

and $x = 25.$

Substituting in (2) the value of x,

$$75 + 4y = 195. \quad (6).$$

Transposing, $4y = 120,$

and $y = 30.$

The elimination in the foregoing was by *addition.*

210. Instead of eliminating y and substituting for x, the reverse might have been done. Thus,

$$5x - 6y = -55, \quad (1)$$

$$3x + 4y = 195. \quad (2)$$

Multiplying (1) by 3 and (2) by 5,

$$15x - 18y = -165, \quad (3)$$

$$15x + 20y = 975. \quad (4)$$

Subtracting (3) from (4),

$$38y = 1140, \tag{5}$$

and $$y = 30.$$

Substituting in (2) the value of y,

$$3x + 120 = 195. \tag{6}$$

Transposing, $$3x = 75,$$

and $$x = 25.$$

Here the elimination was by *subtraction.*

Rule.—I. *Multiply each equation by a quantity that will render the co-efficient of one of the unknown quantities the same in both.*

II. *Subtract or add the resulting equations according as the signs of the equal co-efficients are like or unlike.*

III. *Solve the equation thus found, and substitute the resulting value in one of the given equations.*

NOTE.—Sometimes the work may be shortened by adding or subtracting the given equations as they stand.

Thus, let it be required to find the values of x and y from the equations,

$$x + 13y = 270, \tag{1}$$

$$13x + y = 150. \tag{2}$$

Adding (1) and (2), $$14x + 14y = 420. \tag{3}$$

Dividing (3) by 14, $$x + y = 30. \tag{4}$$

Subtracting (4) from (1), $$12y = 240. \tag{5}$$

Whence, $$y = 20.$$

Subtracting (4) from (2), $$12x = 120. \tag{6}$$

Whence, $$x = 10.$$

EXERCISES.

211. ***Solve by addition or subtraction:***

1. $7x - 2y = 68$; $4x - 5y = 8$.
2. $3x + 2y = 26$; $5x - 2y = 38$.
3. $4y + x = 102$; $y + 4x = 48$.
4. $2x - y = 6$; $4x + 3y = 22$.
5. $2x + 3y = 23$; $5x - 2y = 10$.
6. $3x + 4y = 188$; $2x + 5y = 207$.
7. $2x + 7y = 34$; $5x + 9y = 51$.
8. $x + 4y = 16$; $4x + y = 34$.
9. $12x + 13y = 37$; $17x - 19y = 15$.
10. $2x - 9y = 11$; $x - 4y = 5$.
11. $9x - 4y = 8$; $13x + 7y = 101$.
12. $3y - x = 104$; $5x - 2y = 78$.
13. $7y - 3x = -7$; $11x + 5y = 87$.
14. $5x - 3y = 9$; $5y + 2x = 16$.
15. $2x + 3y = 32$; $11y - 9x = 3$.
16. $x - \frac{2y - x}{23 - x} = 20 - \frac{59 - 2x}{2}$;

    $y + \frac{y - 3}{x - 18} = 30 - \frac{73 - 3y}{3}$.
17. $\frac{6y + 9}{4} + \frac{3y - 5}{4x - 6} = 1\frac{1}{2} + \frac{3y + 4}{2}$;

    $\frac{8y + 7}{10} - \frac{3y - 6x}{2y - 8} = 4 - \frac{9 - 4y}{5}$.

18. $\frac{x}{3y} = 1\frac{1}{3}$; $\frac{1}{5}(2x + 7y) - 1 = \frac{2}{3}(2x - 6y + 1)$.

19. $y - 3x = 1$; $2(3x + y) = 3(y - x) + 1$.

20. $x + \frac{y}{2} = 17$; $\frac{2y - x}{3x + y} = \frac{13}{45}$.

21. $56(1 - x - y) = 5$; $51(y - x) = 19(x + y)$.

22. $\frac{1}{x} + \frac{1}{y} = \frac{8}{9}$; $4x - 3y = 2xy$.

23. $22x - y = 33$; $\frac{y + x}{5x - y} + \frac{5}{8} = 0$.

24. $\frac{y + x}{y - x} = 12$; $\frac{3x - y}{2} - \frac{2y - x}{5} = 7$.

25. $5(x + 1) + 3(2y - 5) = 2$; $4x - 10y = 17$.

26. $\frac{1}{2y} - \frac{1}{3x} = \frac{7}{15}$; $\frac{y - 5x + 4xy}{4xy} = \frac{1}{15}$.

LITERAL EQUATIONS.

212. Either of the preceding methods may be applied in solving literal equations, but the method by addition or subtraction is generally the best.

I. Let it be required to find the values of x and y in the equations,

$$ax - by = c. \quad (1)$$

$$mx + ny = r. \quad (2)$$

Multiplying (1) by n and (2) by b,

$$anx - bny = cn, \tag{3}$$

$$bmx + bny = br. \tag{4}$$

Adding (3) and (4),

$$anx + bmx = cn + br. \tag{5}$$

Factoring, $(an + bm)\,x = cn + br.$

Dividing by co-efficient of x,

$$x = \frac{cn + br}{an + bm}. \tag{6}$$

The value of y may be found by substituting in (1) or in (2) the value of x in (6). It is better, however, to eliminate x precisely as y was eliminated.

$$ax - by = c, \tag{1}$$

$$mx + ny = r. \tag{2}$$

Multiplying (1) by m and (2) by a,

$$amx - bmy = cm, \tag{3}$$

$$amx + any = ar. \tag{4}$$

Subtracting (3) from (4),

$$any + bmy = ar - cm. \tag{5}$$

Whence, $$y = \frac{ar - cm}{an + bm}. \tag{6}$$

EXERCISES.

213. ***Find the values of x and y in the following equations:***

1. $ax + by = c$; $px + qy = r$.
2. $ax - by = m$; $cx + ey = n$.
3. $x + ay = b$; $ax - by = c$.
4. $ax + 2by = c$; $a(3a - x) = b(x + y)$.
5. $ax + by = c$; $bx - ay = d$.
6. $ax - by = 0$; $x + y = c$.
7. $\frac{x}{a} + \frac{y}{b} = 2$; $bx - ay = 0$.
8. $x + y = a + b$; $bx + ay = 2ab$.
9. $\frac{x}{a} + \frac{y}{b} = 1$; $\frac{x}{b} + \frac{y}{a} = 1$.
10. $(a + c)x - by = bc$; $x + y = a + b$.
11. $\frac{x}{a} + \frac{y}{b} = c$; $\frac{x}{b} - \frac{y}{a} = 0$.
12. $\frac{m}{x} + \frac{n}{y} = a$; $\frac{n}{x} + \frac{m}{y} = b$.
13. $x + y = c$; $ax - by = c(a - b)$.
14. $a(x + y) + b(x - y) = 1$;
    $a(x - y) + b(x + y) = 1$.
15. $\frac{x - a}{b} - \frac{b - y}{a} = 0$;
    $\frac{x + y - b}{a} - \frac{a + y - x}{b} = 0$.

16. $(a+b)x-(a-b)y=4ab;$
$(a-b)x+(a+b)y=2(a^2-b^2).$

17. $\frac{x}{a+b}+\frac{y}{a-b}=2a;\quad \frac{x-y}{2ab}=\frac{x+y}{a^2+b^2}.$

18. $(a+h)x+(b-h)y=c;$
$(b+k)x+(a-k)y=c.$

19. $\frac{x}{b+c}+\frac{y}{a+c}=1;\quad \frac{ax+cx}{(a+b)c}=1.$

20. $\frac{x}{a+b}+\frac{y}{a-b}=2a;\quad \frac{x-y}{4ab}=1.$

MISCELLANEOUS EXERCISES.

214. ***Solve by any one of the preceding methods the following:***

1. $3x+5y=55;\quad 5x-y=45.$

2. $5x-11y=19;\quad 7x+y=43.$

3. $3x+2y=118;\quad x+5y=191.$

4. $2x-y=1;\quad x+2y=8.$

5. $3x+y=5;\quad x+y=3.$

6. $x+4y=11;\quad x-y=1.$

7. $x-y=-3;\quad 3x-y=1.$

8. $5x-2y=-1;\quad x+y=4.$

9. $7x+2\frac{1}{2}y=411\frac{1}{2};\quad 39x-14y=-935.9.$

10. $5x-8\frac{1}{3}=7y+21;\quad 2x=5y-6\frac{3}{4}.$

11. $\frac{x}{4} - \frac{y}{2} = 1$; $\frac{x}{12} + \frac{y}{6} = 1$.

12. $32x + 81y = 43$; $28x - 39y = 1$.

13. $96x + 75y = 102$; $92x + 80y = 101$.

14. $2x + y = \frac{1}{2}$; $18x + 22y = -28$.

15. $11x + 9y = 40$; $\frac{3x + y}{8} - \frac{2x - y}{31} = 2$.

16. $5\frac{1}{4}y - 11x = 4y + 117\frac{1}{8}$; $8x + 175 = 2y$.

17. $13x + 7y = 341 + 7\frac{1}{2}y + 43\frac{1}{2}x$; $2x + \frac{1}{2}y = 1$.

18. $a(x - a) + b(y - b) = 0$;
$a(x - y - a) + b(x + y - b) = 0$.

19. $m(x + y) + n(x - y) = 1$;
$m(x - y) + n(x + y) = 1$.

20. $bcx = cy - 2b$;
$b^2y + \frac{a(c^3 - b^3)}{bc} = \frac{2b^3}{c} + c^2x$.

215. When there are three or more equations in the given set, first obtain a set containing one unknown quantity fewer; from that set, another set in which there are two missing, and so on; until finally there is but one equation with one unknown quantity.

I. Required to find the values of x, y, and z in the equations:

$$2x - 5y + z = 3, \quad (1)$$
$$x + 3y - 2z = 16, \quad (2)$$
$$3x - y - 2z = 20. \quad (3)$$

$(2) \times 2 =$ $\quad 2x + 6y - 4z = 32,$

$(1) =$ $\quad 2x - 5y + z = 3,$

Subtracting, $\quad 11y - 5z = 29. \quad (4)$

$(2) \times 3 =$ $\quad 3x + 9y - 6z = 48,$

$(3) =$ $\quad 3x - y - 2z = 20,$

Subtracting, $\quad 10y - 4z = 28. \quad (5)$

$(5) \times 5 =$ $\quad 50y - 20z = 140,$

$(4) \times 4 =$ $\quad 44y - 20z = 116,$

Subtracting, $\quad 6y = 24, \quad (6)$

$y = 4.$

Substituting value of y in (5), $z = 3$.

Substituting values of y and z in (2),

$x = 10.$

II. Required to find the values of x, y, u, and v in the equations,

$$2x + 3y - u + 4v = 41, \quad (1)$$

$$x - 5y + 4u - v = 6, \quad (2)$$

$$3x - y - u + v = 22, \quad (3)$$

$$4x + y + u - 5v = 36. \quad (4)$$

$(1) =$ $\quad 2x + 3y - u + 4v = 41,$

$(2) \times 2 =$ $\quad 2x - 10y + 8u - 2v = 12,$

Subtracting, $\quad 13y - 9u + 6v = 29. \quad (5)$

$(3) =$ $\quad 3x - y - u + v = 22,$

$(2) \times 3 =$ $\quad 3x - 15y + 12u - 3v = 18,$

Subtracting, $\quad 14y - 13u + 4v = 4. \quad (6)$

$(1) \times 2 = \quad 4x + 6y - 2u + 8v = 82,$
$(4) = \quad 4x + y + u - 5v = 36,$
Subtracting, $\quad 5y - 3u + 13v = 46. \quad (7)$

$(7) \times 3 = \quad 15y - 9u + 39v = 138,$
$(5) = \quad 13y - 9u + 6v = 29,$
Subtracting, $\quad 2y + 33v = 109. \quad (8)$

$(7) \times 13 = \quad 65y - 39u + 169v = 598,$
$(6) \times 3 = \quad 42y - 39u + 12v = 12,$
Subtracting, $\quad 23y + 157v = 586. \quad (9)$

$(8) \times 23 = \quad 46y + 759v = 2507,$
$(9) \times 2 = \quad 46y + 314v = 1172,$
Subtracting, $\quad 445v = 1335, \quad (10)$
$v = 3.$

Substituting value of v in (8), $\quad y = 5.$

Substituting values of v and y in (7),
$u = 6.$

Substituting values of v, y, and u in (2),
$x = 10.$

EXERCISES.

216. ***Find the value of the unknown quantities in the following equations:***

1. $2x + 3y + 4z = 20, \quad 4x - 3y - 2z = -8,$
   $3x + 4y + 5z = 26.$
2. $2x - 3y + z = 1, \quad 3x - 5y + 4z = 3,$
   $4x + 2y - 3z = 13.$
3. $x + 3y + 2z = 11, \quad 2x + y + 3z = 14,$
   $3x + 2y + z = 11.$

4. $5x - 6y + 4z = 15$, $7x + 4y - 3z = 19$,
$2x + y + 6z = 46$.

5. $4x - 5y + z = 6$, $7x - 11y + 2z = 9$,
$x + y + 3z = 12$.

6. $3x - y + z = 17$, $5x + 3y - 2z = 10$,
$7x + 4y - 5z = 3$.

7. $x + y + z = 5$, $3x - 5y + 7z = 75$,
$9x + 11y + 10 = 0$.

8. $\frac{x}{2} + \frac{y}{4} + \frac{z}{6} = 3$, $\frac{x}{4} + y - \frac{z}{12} = 4$,
$\frac{x}{5} + y + \frac{z}{10} = 5$.

9. $\frac{4x}{5} - \frac{y-3}{2} = 2$, $\frac{x+y+z}{7} = 3$,
$\frac{x+y}{4} + \frac{z}{9} = 4$.

10. $x + y + z = 29\frac{1}{4}$, $x + y - z = 18\frac{1}{4}$,
$x - y + z = 13\frac{3}{4}$.

11. $3x + 5y = 161$, $7x + 2z = 209$, $2y + z = 89$.

12. $y + \frac{1}{2}x = 41$, $x + \frac{1}{4}z = 20\frac{1}{2}$, $y + \frac{1}{2}z = 34$.

13. $53 - \frac{1}{2}x - \frac{1}{2}z = y - 109$, $\frac{1}{4}x + \frac{1}{3}y = 26$,
$5y = 4z$.

14. $2x - \frac{3}{4}y = 93 - \frac{1}{2}x - \frac{1}{4}y$, $7x - 5z = x + y - 86$.
$\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 58$.

15. $\frac{1}{x} + \frac{1}{y} = a$, $\frac{1}{x} + \frac{1}{z} = b$, $\frac{1}{y} + \frac{1}{z} = c$.

16. $x - 9y + 3z - 10u = 21,$
$2x + 7y - z - u = 683,$
$3x + y + 5z + 2u = 195,$
$4x - 6y - 2z - 9u = 516.$

17. $x + y + z + u = 1, \quad 16x + 8y + 4z + 2u = 9,$
$81x + 27y + 9z + 3u = 36,$
$256x + 64y + 16z + 4u = 100.$

18. $\frac{x}{3} + \frac{y}{5} + \frac{2z}{7} = 58, \quad \frac{5x}{4} + \frac{y}{6} + \frac{z}{3} = 76,$

$\frac{x}{2} + \frac{3z}{8} + \frac{u}{5} = 79, \quad y + z + u = 248.$

19. $3x - 4y + 3z - 4u + 8 = 0,$
$2x - 3y + 5z - 6u + 4 = 0,$
$x + 2y - 3z + u - 2 = 0,$
$4x - y - 2z - u + 1 = 0.$

20. $5x + 4y - 3z - 2u - 1 = 0,$
$x + y + 3z - 4u + 2 = 0,$
$3x - y + 2z - 3u + 5 = 0,$
$5x - 2y + z - 2u + 1 = 0.$

21. $x + y + z + 9 = 4u, \quad 5x - 3y + 5z + 16 = 7u,$
$3x - 4y + 4z + 23 = 2u,$
$4x - 3y - 3z + 11 = u.$

22. $4x + 2z = 3y + 5u, \quad 3x + z = 2y + 2u + 12,$
$x - y - z + 3u = 4, \quad 2x - 3y + 3z - 4u = 2.$

23. $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 62, \quad \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 47,$

$\frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 38.$

24. $\frac{3}{x} - \frac{4}{5y} + \frac{1}{z} = 7\frac{1}{5}$, $\frac{1}{3x} + \frac{1}{2y} + \frac{2}{z} = 10\frac{1}{6}$,
$\frac{4}{5x} - \frac{1}{2y} + \frac{4}{z} = 16\frac{1}{10}$.

25. $x + 2z + 3v + 4t = 14$,
$-x + 3y + 2z + 5v + 3t = 64\frac{1}{2}$,
$x + 6y - z + 8v = 70$,
$-x + y - z + 4v - t = 22\frac{1}{2}$,
$x - 3y + 3z - 2v + 4t = -24$.

26. $ax + by + cz = 3$, $ax + by - cz = 1$,
$ax - by + cz = 1$.

PROBLEMS LEADING TO EQUATIONS WITH TWO OR MORE UNKNOWN QUANTITIES.

217. I. There are two numbers such that the sum of $\frac{1}{3}$ of the first and $\frac{1}{2}$ of the second is 50, and the sum of $\frac{1}{5}$ of the first and $\frac{1}{7}$ of the second is 19. What are the numbers?

Solution.

Let x = the first number, and y = the second number.

Then, $$\frac{x}{3} + \frac{y}{2} = 50, \qquad (1)$$

and $$\frac{x}{5} + \frac{y}{7} = 19. \qquad (2)$$

(1) × 3 = $$x + \frac{3y}{2} = 150, \qquad (3)$$

(2) × 5 = $$x + \frac{5y}{7} = 95. \qquad (4)$$

$(4) - (3) =$ $\frac{11y}{14} = 55.$ (5)

Reducing, $\frac{y}{14} = 5.$

Whence, $y = 70.$ (6)

Substituting in (1), $\frac{x}{3} + 35 = 50.$ (7)

Reducing and transposing, $x = 45.$ (8)

1. There are two numbers such that 3 times the first plus $\frac{1}{2}$ of the second is 108, and $\frac{1}{2}$ of the first plus $\frac{2}{3}$ of the second is 32. What are the numbers?

2. There is a fraction such that 3 added to its numerator makes its value $\frac{1}{2}$, and 1 subtracted from its denominator makes its value $\frac{1}{3}$. Find the fraction.

3. Find two numbers such that $\frac{1}{2}$ of the first minus $\frac{1}{3}$ of the second is 19, and $\frac{1}{3}$ of the first minus $\frac{1}{4}$ of the second is 8.

II. There is a number consisting of two digits. The number is equal to three times the sum of its digits, and if 45 be added to the number the digits interchange their places. Find the number.

Solution.

Let x = tens' digit, and y = units' digit.

Then $10x + y$ = the number, and $10y + x$ = the number with interchanged digits.

Now, $3(x + y) = 10x + y,$ (1)

and $10x + y + 45 = 10y + x.$ (2)

From (1), $$2y = 7x,$$

and $$y = \frac{7x}{2}. \qquad (3)$$

From (2), $$9x - 9y = -45,$$

and $$x - y = -5. \qquad (4)$$

Substituting in (4) the value of y in (3),

$$x - \frac{7x}{2} = -5.$$

Whence, $$x = 2.$$

Substituting in (4) the value of x,

$$y = 7.$$

Hence, the number is 27.

4. Find a number equal to seven times the sum of its digits, and such that if 27 be subtracted from it the digits interchange their places.

5. The units' digit of a number of two places is twice the tens' digit; if 6 be subtracted from double the number, the remainder will be the number with its digits interchanged. Find the number.

6. A number consisting of two figures is equal to four times the sum of its digits; if 18 be added to the number, its digits will be interchanged. What is the number?

7. A certain fraction becomes equal to $\frac{1}{2}$ when its terms are both increased by 1, and to $\frac{1}{3}$ when its numerator is decreased by 2 and its denominator by 3. What is the fraction?

8. Find two numbers such that $\frac{2}{3}$ of the first plus $\frac{1}{4}$ of the second shall be equal to the second plus 4; and three times the first shall exceed twice the second by 8.

9. A's wages for 2 days, B's for 6 days, and C's for 3 days amount to $28; or A's for 1 day, B's for 2 days, and C's for 4 days amount to $19; or, again, A's for 5 days, B's for 4 days, and C's for 2 days amount to $26. Find the daily wages of each.

10. Find three numbers such that one half the first, one third the second, and one fifth the third shall together be equal to 39; one third the first, one fourth the second, and one half the third equal to 42; or, one fifth the first, one sixth the second, and one fourth the third equal to 24.

11. The value of a certain fraction is $\frac{1}{4}$ when 4 is added to its numerator, and $\frac{1}{2}$ when its denominator is diminished by 2. What is the fraction?

12. A sum of money was divided equally among a certain number of persons; had there been 10 more, each would have received $2 less, and had there been 5 fewer, each would have received $2 more than he did. How many persons were there, and what did each receive?

13. A and B each has a sum of money. B pays A $4, and then has twice as much as A. If A had paid B $6, he would have had $\frac{1}{3}$ as much as B. How much had each?

14. A and B can together earn $49 in 10 days; A and C $51 in 10 days; B and C can earn $22 in 4 days. Find what each can earn in a day.

15. The sum of two numbers is 230. One eleventh of the first number is 10 more than one thirteenth of the second number. What are the numbers?

16. A certain garrison numbering 2744 men is composed of cavalry, artillery, and infantry. Half the number of infantry increased by 245 equals the sum of 3 times the number of the artillery and 6 times the number of cavalry; and 8 times the number of the artillery and cavalry together equals

the number of infantry diminished by 98. How many are there of each kind?

17. If A's money were increased by $145, he would have twice as much as B. If B's money were increased by $37, he would have $\frac{4}{5}$ as much as A. How much has each?

18. A miller has three kinds of wheat, worth respectively $1.05, $1.10, and $1.19 per bushel. Wishing to make a mixture of 100 bushels worth $1.11 a bushel, he first takes 25 bushels worth $1.05. How many bushels each of the other kinds must he put into the mixture?

19. If my parlor were 6 feet longer and 2 feet wider, it would take 20 square yards more of carpet to cover it; but if it were 2 feet longer and 9 feet wider, it would take 36 square yards more. What are its dimensions?

20. Two pipes, A and B, will fill a cistern in 70 minutes, A and C will fill it in 84 minutes, and B and C in 140 minutes. How long will it take each alone to fill the cistern?

21. A and B together have $13500. A invests $\frac{2}{3}$ of his money and B $\frac{4}{5}$ of his, after which they together have $8250 left. How much has each at first?

22. A drover bought 10 oxen, 120 sheep, and 45 lambs for $1280. He paid the same for 3 sheep and 7 lambs as for one ox. The cost of 2 oxen, 6 sheep, and 8 lambs was just $\frac{1}{10}$ of the cost of all the animals. How much did each cost?

23. A worked 10 days, B 4 days, and C 3 days, and their wages amounted to $29; at another time, A worked 9 days, B 8 days, and C 6 days, and their wages amounted to $36; a third time, A worked 7 days, B 6 days, and C 4 days, and their wages amounted to $27. How much did each earn in one day?

24. A farmer sells to one person 9 horses and 7 cows for $2620. To another person, and at the same price per head, he

sells 3 horses and 4 cows for $1015. What was the price of each?

25. Three brothers, A, B, and C, bought a farm for $5000. A could pay for it alone by borrowing $\frac{1}{3}$ of B's money and $\frac{1}{2}$ of C's money. B could pay for it by borrowing $\frac{1}{4}$ of A's and $\frac{1}{4}$ of C's money. C could pay for it by borrowing $\frac{1}{4}$ of A's and $\frac{1}{3}$ of B's money. How much money has each?

26. Find what each of three persons, A, B, and C, is worth, from knowing, 1st, that what A is worth added to 3 times what B and C are worth, makes 4700 dollars; 2d, that what B is worth added to 4 times what A and C are worth makes 5800 dollars; 3d, that what C is worth added to 5 times what A and B are worth makes 6300 dollars.

27. A man has three-cent pieces, dimes, and twenty-five-cent pieces. He has of the second and third kinds together 310 coins. If his dimes were three-cent pieces and his three-cent pieces dimes, the two kinds would amount to $17.50. As it is, he has $43. Find how many coins he has of each kind.

28. Half of A's money is $1500 more than the sixth of B's and C's together; a fourth of B's is equal to the excess of A's above C's; and the sum of A's and C's lacks $1000 of being three times B's. Find the amount each has.

29. After A had paid $374 to B, and $50 to C, they each had the same sum. Before making the payments, A lacked only $228 of having as much as B and C together. How much had each at first?

30. A, B, and C have together $1820. If B gives A $200, A will have $160 more than B; but if B gets $70 from C, they two will have the same sum. How much has each?

31. Three towns are at the three angles of a triangle. From the first to the second through the third is 82 miles; from the first to the third through the second is 97 miles;

from the second to the third through the first is 89 miles. Find the direct distances between the towns.

32. Three persons incur a debt which neither can pay in full. A says to B, "Give me $\frac{1}{5}$ of your money and I can pay all the bill." B says to C, "Give me $\frac{1}{8}$ of your money and I can pay all." C says to A, "Give me half of your money, and, although I have only \$4, I can pay all." Find the money of A and of B, and the bill.

33. A cistern has three pipes. The first and second running together can fill it in 100 minutes, the first and third in 108 minutes, the second and third in $122\frac{8}{11}$ minutes. How long will it take each, and how long all together, to fill it?

34. A grocer mixes three kinds of coffee. He can sell a mixture containing 3 lb. of the first kind, 11 lb. of the second, and 4 lb. of the third, at 22 cts. per lb.; or one composed of 9 lb. of the first, 6 lb. of the second, and 3 lb. of the third, at 21 cts. per lb.; or, finally, he can sell at 20 cts. per lb. a mixture of 5 lb. of the first kind, 3 lb. of the second, and 6 lb. of the third. Find the cost per lb. of each kind.

35. A number consists of three figures. The hundred's figure is equal to the sum of the other two; if the number be divided by the sum of its digits, the quotient will be 54; and 693 subtracted from the number will interchange its digits. Find the number.

36. If \$20 be taken from the first and put into the second of three purses, the second will contain 4 times as much as remains in the first. If \$60 of what is now in the second be put in the third, the third will contain just twice what is in the first and second together. If \$40 be now taken from the third and put into the first, there will be one half as much in the first as in the third. How much is in each at first?

37. On the road from A to B there are 24 miles up hill, 36 miles down hill, and 12 miles level. A courier requires 25 hours to go; and 28 hours to return; but if, while going from A to B, there had been 12 miles up hill and 24 miles level, he could have gone in 23 hours. Find his rate per hour up hill, down hill, and on the level.

38. Two persons, A and B, can do a piece of work in 16 days. They work together 4 days, after which B finishes the work in 36 days more. In what time can each do it separately?

39. The sum of three numbers is 26. The sum of the first, twice the second, and four times the third is 73. The sum of the second, three times the first, and five times the third is 88. What are the numbers?

40. A and B together can do a certain work in 12 days; B and C can do it in 20 days; A and C can do it in 15 days. In what time can all do the work together, and in what time can each do it separately?

41. I have two horses of unequal value, and two saddles worth $50 and $10 respectively. The better saddle and the first horse are together worth $10 more than the worse saddle and the second horse. But the worse saddle and the first horse are together worth only $30 more than half as much as the better saddle and the second horse. What is the value of each horse?

42. A man at his death leaves $29250 to be divided among his family—three sons, four daughters, and their mother. A son and a daughter together receive twice as much as the mother. Two sons together are to receive $2250 more than the mother and one daughter. Find the shares of each.

INVOLUTION.

218. A **power** of a quantity is the product obtained by using the quantity two or more times as a factor.

219. Involution is the process of finding any required power of a quantity.

220. A **perfect power** is a quantity that can be resolved into two or more equal factors.

Thus, 4 is a **perfect square** or **second power**, since $4 = 2 \times 2$.

64 is a **perfect cube** or **third power**, since $64 = 4 \times 4 \times 4$.

221. The **exponent** of a power is a small figure or letter written above and at the right of a quantity. When it is a positive integer, it shows how many times the quantity is to be taken as a factor.

PRINCIPLES.—I. *Any power of a positive quantity is positive.*

II. *The even powers of a negative quantity are positive, and the odd powers are negative.*

Thus, $(+x)^2 = (+x) \times (+x) = +x^2$.

$(+x)^3 = (+x) \times (+x) \times (+x) = +x^3$.

$(-x)^2 = (-x) \times (-x) = +x^2$.

$(-x)^3 = (-x) \times (-x) \times (-x) = -x^3$.

222. To find any required power of a monomial.

I. Required to find the cube of $5a^2b^3x$.

$$(5a^2b^3x)^3 = 5a^2b^3x \times 5a^2b^3x \times 5a^2b^3x = 125a^6b^9x^3.$$

II. Required to find the cube of $-3x^2yz^3$.

$$(-3x^2yz^3)^3 = (-3x^2yz^3) \times (-3x^2yz^3) \times (-3x^2yz^3)$$
$$= -27x^6y^3z^9.$$

NOTE.—Fractional quantities are raised to any given power by raising both numerator and denominator to the required power. Thus,

$$\left(-\frac{2a^3b}{3xy^3}\right)^3 = \left(-\frac{2a^3b}{3xy^3}\right) \times \left(-\frac{2a^3b}{3xy^3}\right) \times \left(-\frac{2a^3b}{3xy^3}\right) = -\frac{8a^9b^3}{27x^3y^6}.$$

Rule.—I. *Raise the numerical co-efficient to the required power. To this power annex the literal factors, each with an exponent equal to the product of its original exponent multiplied by the exponent of the power.*

II. *Make the result positive for all powers of positive quantities, and for all even powers of negative quantities. Make the result negative for all odd powers of negative quantities.*

EXERCISES.

223. ***Write the values of the following powers:***

1. $(abx^2)^3$.
2. $(-x^2y)^3$.
3. $(-x^4yz^2)^4$.
4. $(3a^3x^2)^5$.
5. $(-2m^3n)^5$.
6. $(-3mn^2x)^4$.
7. $(5m^4n^3y^2)^3$.
8. $(-7a^2mn^2)^3$.
9. $(16a^4b^5m^9)^2$.
10. $(10x^2y^3z^4)^3$.
11. $(-5a^2b^3c^3d^4)^4$.
12. $(-4a^5m^3n^4x^2)^5$.

13. Write the square of $\frac{1}{3}m^3$. Of $\frac{1}{3}m^2n$. Of $4a^2b^2c^3$.

14. Write the cube of $\frac{1}{2}a^2b$. Of $\frac{1}{4}x^3y^2z$. Of $5am^4x^2$.

15. Write the square of $-10a^3bx^2$. Of $-15xyz^4$.

16. Write the cube of $-\frac{5}{12}x^2yz^3$. Of $\frac{2}{11}m^4n^2y^3$.

17. Write the fifth power of $\frac{1}{2}am^2n^2$. Of $-\frac{1}{3}x^2yz^3$.

18. Write the square of $\frac{1}{4}p^2q$. Of $\frac{1}{5}ax^2y$. Of $\frac{1}{3}am^2$.

19. Write the fourth power of $\frac{2}{3}a^2m^3n$. Of $\frac{3}{4}ab^2c^2x$.

20. Write the cube of $\frac{1}{3}x^3yz^4$. Of $\frac{2}{10}m^4n^3y^2$.

21. Write the square of $-\frac{2x^2}{3y}$. Of $\frac{3ax^2}{2y^3}$. Of $-\frac{x^2y^2}{3az}$.

22. Write the cube of $\frac{4a^2x^4}{5y^2z}$. Of $\frac{-3a^2xy}{8m^2n^3}$.

23. Write the fourth power of $\frac{-3a^2b^4c}{-5xy^2z^2}$. Of $\frac{2m^3n^3}{-3x^2y^3z^3}$.

224. The method of finding the square of a polynomia. has already been explained in Art. **92.** Some additional exercises are given below.

EXERCISES.

225. ***Write the values of the following indicated squares:***

1. $(x + y + z)^2$.

2. $(a + 2b + 3c)^2$.

3. $(2m - 3n + 5y)^2$.

4. $(3x - 2y - 4z)^2$.

5. $(9x - 3y + z)^2$.

6. $(1 + 2x + x^2)^2$.

7. $(a^2 + b^2 - c^2 + d)^2$.

8. $(4m + m^2 + 1)^2$.

9. $(a + b + c - d)^2$.

10. $(x^3 - 3x^2 + 2x - 5)^2$.

11. $(1 + 3a + 3a^2 + a^3)^2$.
12. $(1 - 3x + 3x^2 - x^3)^2$.
13. $(\frac{1}{4}x - 1\frac{1}{2}y + 1\frac{1}{4}z)^2$.
14. $(2x^2 + x - 7)^2$.
15. $(1 + x + x^2)^2$.
16. $(\frac{1}{2}a + a^2b + \frac{1}{2}b)^2$.
17. $(m^2 + \frac{1}{2}m + \frac{1}{2})^2$.
18. $(5x + 3y^2 - .25z)^2$.

226. To find the cube of a binomial.

I. Required to find the value of $(x+y)^3$. Also of $(x-y)^3$.

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2\\ &\quad\ x + y\\ &\overline{\quad x^3 + 2x^2y + \ \ xy^2}\\ &\quad\ \ x^2y + 2xy^2 + y^3\\ &\overline{(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3}\end{aligned}$$

$$\begin{aligned}(x-y)^2 &= x^2 - 2xy + y^2\\ &\quad\ x - y\\ &\overline{\quad x^3 - 2x^2y + \ \ xy^2}\\ &\quad\ -\ x^2y + 2xy^2 - y^3\\ &\overline{(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3}\end{aligned}$$

II. Required to find the cube of $2x - 3y$.

$$\begin{aligned}(2x-3y)^2 &= 4x^2 - 12xy + 9y^2\\ &\quad\ 2x - 3y\\ &\overline{\quad 8x^3 - 24x^2y + 18xy^2}\\ &\quad\ -\ 12x^2y + 36xy^2 - 27y^3\\ &\overline{(2x-3y)^3 = 8x^3 - 36x^2y + 54xy^2 - 27y^3}\end{aligned}$$

Rule.—*Write the cube of the first term; three times the square of the first term multiplied by the second term; three times the first term multiplied by the square of the second term; and the cube of the second term.*

EXERCISES.

227. ***Write the values of the following indicated cubes:***

1. $(a + b)^3$.
2. $(2x + y)^3$.
3. $(3m - 2n)^3$.
4. $(2a - 3b)^3$.
5. $(3a + 4b)^3$.
6. $(4a - 3y)^3$.
7. $(m - 2n)^3$.
8. $(a + 3b)^3$.
9. $(2a - b)^3$.
10. $(1 + x)^3$.
11. $(1 - x^2)^3$.
12. $(x - 2y)^3$.
13. $(a + 5)^3$.
14. $(\frac{1}{2}x - \frac{1}{2}y)^3$.
15. $(1 - mn)^3$.
16. $(2a + c)^3$.
17. $(\frac{1}{4}x + \frac{1}{2}y)^3$.
18. $(4x + y)^3$.
19. $(\frac{2}{3}a + b)^3$.
20. $(x - \frac{1}{3}y)^3$.
21. $(a^2 + b^2)^3$.
22. $(x^2 - 2y)^3$.
23. $(m - n^2)^3$.
24. $(3a^2 + b^2)^3$.
25. $(x^3 + 2x^2y)^3$.
26. $(x^2 + xy)^3$.
27. $(2x^2 - 3xy)^3$.
28. $(3ab - 2b^2)^3$.

228. To find the cube of a trinomial.

Required to find the cube of $a^2 - 2a + 5$.

$$\begin{aligned}(a^2 - 2a + 5)^3 &= \{(a^2 - 2a) + 5\}^3 \\ &= (a^2 - 2a)^3 + [3(a^2 - 2a)^2 \times 5] \\ &\qquad + [3(a^2 - 2a) \times 5^2] + 5^3 \\ &= (a^6 - 6a^5 + 12a^4 - 8a^3) + (15a^4 - 60a^3 + 60a^2) \\ &\qquad + (75a^2 - 150a) + 125 \\ &= a^6 - 6a^5 + 27a^4 - 68a^3 + 135a^2 - 150a + 125.\end{aligned}$$

Rule.—I. *By means of parentheses write the trinomial in the form of a binomial.*

II. *Indicate the cube of the first term of this binomial, three times the square of the first term multiplied by the second term, three times the first term multiplied by the square of the second term, and the cube of the second term.*

III. *Perform the indicated operations; unite similar terms; arrange the terms, and connect them by the proper signs.*

EXERCISES.

229. ***Write the cubes of the following, and arrange the results:***

1. $a + b + c$.
2. $x - y + z$.
3. $m - n - r$.
4. $a + 2b + c$.
5. $x - 2y + z$.
6. $x + 2y + 3z$.
7. $a - 2b - 3c$.
8. $2a - m + 2n$.
9. $2p - 3q - 5r$.
10. $a - 5y + z$.
11. $a^2 + x^2 - y^2$.
12. $3x^2 + y^2 + 2z$.
13. $5x - 2x^2 + x^3$.
14. $a^2 + ab + b^2$.
15. $x^2 - xy + y^2$.
16. $x^2 + 2x + 3$.
17. $2x^2 + 3xy + y^2$.
18. $3a^2 - 2ay + 5y^2$.
19. $m^2 - 5mn + 3n^2$.
20. $3a^4 + 5a^2y^2 - 2y^4$.
21. $x^4 - x^2y^2 + y^4$.
22. $1 + p^2 - p^4$.
23. $2x^2 - 3xy + y^2$.
24. $x^4 - 3x^2y - y^3$.
25. $x^3 - 2xy - y^2$.
26. $x^3 - 2x^2y^2 + y^3$.
27. $x^2 - 5x + 6$.
28. $x^4 - 9x^2y + 8y^2$.

230. To find any power of a binomial.

In Art. **81**, the method of writing by inspection any power of a binomial of the form $x \pm y$ has been explained. It is there shown that

$$(x \pm y)^2 = x^2 \pm 2xy + y^2.$$

$$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3.$$

$$(x \pm y)^4 = x^4 \pm 4x^3y + 6x^2y^2 \pm 4xy^3 + y^4.$$

$$(x \pm y)^5 = x^5 \pm 5x^4y + 10x^3y^2 \pm 10x^2y^3 + 5xy^4 \pm y^5.$$

The pupil should become thoroughly familiar with the co-efficients and signs of the different powers of $x \pm y$. They are,

1. For the second power, $1 \pm 2 + 1$.
2. For the third power, $1 \pm 3 + 3 \pm 1$.
3. For the fourth power, $1 \pm 4 + 6 \pm 4 + 1$.
4. For the fifth power, $1 \pm 5 + 10 \pm 10 + 5 \pm 1$.

231. The method explained in Art. **81** is applicable where, instead of x and y, terms are used whose co-efficients and exponents are other numbers than 1.

Thus, $(2a^2 + 3b)^2 = (2a^2)^2 + 2\,(2a^2)\,(3b) + (3b)^2$
$= 4a^4 + 12a^2b + 9b^2.$

$(3x - 5y)^2 = (3x)^2 - 2\,(3x)\,(5y) + (5y)^2$
$= 9x^2 - 30xy + 25y^2.$

$(2m^3 - 3n)^3 = (2m^3)^3 - 3\,(2m^3)^2\,(3n)$
$+ 3\,(2m^3)\,(3n)^2 - (3n)^3$
$= 8m^6 - 36m^4n + 54m^2n^2 - 27n^3.$

$(\frac{2}{3}x + \frac{3}{4}y)^4 = (\frac{2}{3}x)^4 + 4\,(\frac{2}{3}x)^3\,(\frac{3}{4}y) + 6\,(\frac{2}{3}x)^2\,(\frac{3}{4}y)^2$
$+ 4\,(\frac{2}{3}x)\,(\frac{3}{4}y)^3 + (\frac{3}{4}y)^4$
$= \frac{16}{81}x^4 + \frac{8}{9}x^3y + \frac{3}{2}x^2y^2 + \frac{9}{8}xy^3 + \frac{81}{256}y^4.$

Rule.—I. *Indicate each term of the required pow precisely as if the binomial to be involved were* $x \pm y$.

II. *Perform the operation indicated in each term, a reduce to simplest form.*

III. *Arrange the resulting terms and unite them the proper signs.*

EXERCISES.

232. ***Write the values of the following indicated power***

1. $(a + x)^2$.
2. $(m - n)^2$.
3. $(a + b)^3$.
4. $(p + q)^4$.
5. $(2a + x)^2$.
6. $(x - 2y)^3$.
7. $(2a + 3m)^3$.
8. $(3x + 5)^3$.
9. $(2x + 5y)^3$.
10. $(5x + 1)^4$.
11. $(1 + x)^4$.
12. $(a + b)^5$.
13. $(a - b)^5$.
14. $(2x + y)^5$.
15. $(x - 2y)^5$.
16. $(3x + y)^5$.
17. $(m + 5n)^5$.
18. $(2p - 3q)^5$.
19. $(a + 2b)^4$.
20. $(2x + y)^4$.
21. $(\frac{3}{4}x + y)^3$.
22. $(x - \frac{1}{2}y)^3$.
23. $(\frac{2}{3}a - \frac{3}{4}x)^3$.
24. $(a^2 + b^2)^3$.
25. $(x + y^2)^3$.
26. $(\frac{1}{2}m - n^2)^4$.
27. $(ax + 1)^5$.
28. $(3a + xy)^3$.
29. $(\frac{3}{2}x + \frac{2}{3}y)^4$.
30. $(\frac{2}{3}x - \frac{3}{4}y)^4$.

EVOLUTION.

233. A **root** of a quantity is one of two or more equal factors that will, if multiplied together, produce the quantity.

234. The **square root** of a quantity is one of *two* equal factors whose product equals the quantity; the **cube root,** one of *three* equal factors, etc.

235. Evolution is the process of finding any required root of a quantity.

236. The **radical sign,** $\sqrt{\ }$, is placed before and over a quantity to indicate evolution.

237. The **index** of a root is a small figure or letter placed before the radical sign to indicate the root required.

Thus, $\sqrt[3]{64}$ indicates the cube root of 64.

$\sqrt[4]{a+b}$ " " fourth root of $a+b$.

$\sqrt[n]{x+y}$ " " n^{th} root of $x+y$.

238. When no index is written, the *square root* is meant.

Thus, $\sqrt{x-y}$ indicates the square root of $x-y$.

239. Since $(+a)^2 = +a^2$, and $(+a)^4 = +a^4$,

also, since $(-a)^2 = +a^2$, and $(-a)^4 = +a^4$,

it follows that $\sqrt{+a^2} = \pm a$, and $\sqrt[4]{+a^4} = \pm a$.

Hence, *The* **even roots** *of a positive quantity have the double sign; that is, they are either positive or negative.*

240. Since $(-a)^3 = -a^3$, and $(-a)^5 = -a^5$,

it follows that $\sqrt[3]{-a^3} = -a$, and $\sqrt[5]{-a^5} = -a$.

Hence, *The* **odd roots** *of a negative quantity are negative.*

241. Since no *even* power of a quantity can be negative, it follows that

An **even root** *of a negative quantity is impossible.*

242. An **imaginary quantity** is an indicated even root of a negative quantity.

Thus, $\sqrt{-a^2}$, $\sqrt{-5x}$, are imaginary quantities.

243. To find any required root of a monomial.

I. Required to find the square root of $x^4y^2z^6$.

Since the square root of a quantity is one of two equal factors whose product equals the quantity,

$$\sqrt{x^4y^2z^6} = \sqrt{x^{\frac{4}{2}}y^{\frac{2}{2}}z^{\frac{6}{2}} \times x^{\frac{4}{2}}y^{\frac{2}{2}}z^{\frac{6}{2}}} = \pm x^2yz^3.$$

II. Required to find the cube root of $-8a^3b^6c^9$.

$$\sqrt[3]{-8a^3b^6c^9} = -2a^{\frac{3}{3}}b^{\frac{6}{3}}c^{\frac{9}{3}} = -2ab^2c^3.$$

III. Required to find the fourth root of $\frac{16a^4b^8}{81x^8y^{12}}$.

$$\sqrt[4]{\frac{16a^4b^8}{81x^8y^{12}}} = \pm\frac{2ab^2}{3x^2y^3}.$$

Rule.—I. *Take the required root of the numerical coefficient, and divide the exponent of each literal factor by the index of the root.*

II. *Give the double sign* $\pm$ *to an even root of a positive quantity, and to an odd root of a quantity give the sign of the quantity itself.*

EXERCISES.

244. ***Find the values of the following indicated roots:***

1. $\sqrt{25a^2b^4}$.
2. $\sqrt{36x^4y^6}$.
3. $\sqrt{144m^4n^6}$.
4. $\sqrt{81a^4b^6c^{12}}$.
5. $\sqrt[3]{125x^3y^6z^9}$.
6. $\sqrt[4]{81x^4y^8z^{12}}$.
7. $\sqrt[5]{32p^5q^{10}r^{15}}$.
8. $\sqrt[4]{256a^8b^4c^{12}}$.
9. $\sqrt[3]{-27m^6n^3}$.
10. $\sqrt[5]{-32a^5m^{10}x^{15}}$.
11. $\sqrt[4]{625a^8b^{12}c^4}$.
12. $\sqrt[5]{243a^{10}b^{15}c^{20}}$.
13. $\sqrt[3]{512x^6y^3z^{15}}$.
14. $\sqrt[3]{1331a^3b^6x^9}$.
15. $\sqrt[5]{-243a^5x^{10}y^{20}}$.
16. $\sqrt[3]{-125m^6n^9p^{12}}$.
17. $\sqrt{\frac{64}{81}a^2b^4c^6}$.
18. $\sqrt[3]{\frac{27}{64}a^3b^6c^9}$.
19. $\sqrt[3]{-\frac{64a^3x^6y^3}{125m^3n^{12}r^6}}$.
20. $\sqrt[3]{-\frac{125}{216}a^6b^3x^{12}}$.
21. $\sqrt[3]{\frac{343}{1000}a^3b^{12}c^9}$.
22. $\sqrt[4]{\frac{16}{625}x^8y^{12}z^{16}}$.
23. $\sqrt[5]{\frac{32}{243}m^{20}n^{10}r^{15}}$.
24. $\sqrt[3]{\frac{216a^3b^6c^{15}}{125x^6y^9z^{12}}}$.

245. To find the square root of a polynomial.

Evolution consists in reversing the process of involutio
To show this, a polynomial is chosen with whose compositi
the pupil is familiar.

I. Required to find the square root of $x^2 + 2xy + y^2$

Process.

$$\begin{array}{r|l} x^2 + 2xy + y^2 & x + y \\ \hline x^2 & \\ \hline 2x + y \;\Big|\; 2xy + y^2 & \\ \quad 2xy + y^2 & \\ \hline \end{array}$$

Explanation.—In squaring $x + y$, the first term of the result is x^2; hence, the first term of the square root of $x^2 + 2xy + y^2$ is found by taking the square root of x^2. The middle term of the square of $x+y$ is $2xy$, which is twice the first term of the root multiplied by t second; hence, the second term of the square root is found by dividi $2xy$ by twice the first term of the root, giving y. After taking away the square of the first term of the root, the remainder is $2xy + y^2$, $(2x + y)\,y$. Hence, the operation is completed by adding y to $2x$ a multiplying the result by y.

II. Required to find the square root of $x^4 - 30x - 6$ $+ 19x^2 + 25$.

Process.

$$\begin{array}{r|l} x^4 - 6x^3 + 19x^2 - 30x + 25 & x^2 - 3x + 5 \\ \hline x^4 & \\ \hline 2x^2 - 3x \;\Big|\; -6x^3 + 19x^2 & \\ \quad -6x^3 + 9x^2 & \\ \hline 2x^2 - 6x + 5 \;\Big|\; 10x^2 - 30x + 25 & \\ \quad 10x^2 - 30x + 25 & \\ \hline \end{array}$$

Explanation.—The polynomial is first arranged according to th descending powers of x. The square root of x^4 is found, giving x^2 This is squared, and the result subtracted from the polynomial, and th next two terms are brought down. Taking twice the part of the roo already found, and dividing $-6x^3$ by it, gives $-3x$ as the second ter of the root. Uniting this with its sign to $2x^2$, the trial divisor, multiply

ing the result by $-3x$, subtracting, and bringing down the remaining terms of the polynomial, gives $10x^2 - 30x + 25$. Doubling the terms of the root thus far found, for a trial divisor, and dividing $10x^2$ by the first term, gives 5. Uniting this with its sign to $2x^2 - 6x$, the trial divisor, and multiplying the result by 5, completes the operation.

Rule.—I. *Arrange the terms according to the ascending or descending powers of the leading letter. For the first term of the required root, write the square root of the first term of the polynomial and subtract its square from the polynomial.*

II. *Taking twice the root thus found as a trial divisor, divide the first term of the remainder by it, writing the quotient as the second term of the root and also annexing it to the trial divisor. Multiply the divisor thus completed by the second term of the root and subtract the product from the remainder of the polynomial.*

III. *Proceed in this manner until there is no remainder, or until the remainder is not divisible by the trial divisor. In the latter case there is no exact root.*

EXERCISES.

246. ***Find the square roots of the following:***

1. $a^2 + 2ab + b^2$.
2. $a^2 + 6a + 9$.
3. $x^2 - 2xy + y^2$.
4. $m^2 - 2mn + n^2$.
5. $a^2 + 4ab + 4b^2$.
6. $m^2 + 6mn + 9n^2$.
7. $a^2 - 8ab + 16b^2$.
8. $x^2 + 14xy + 49y^2$.
9. $a^2x^2 + 2ax + 1$.
10. $a^2b^2 - 6ab + 9$.
11. $1 + 6mn + 9m^2n^2$.
12. $4x^2 - 12xy + 9y^2$.

11.	$(1 + 3a + 3a^2 + a^3)^2$.	*15.*	$(1 + x + x^2)^2$.
12.	$(1 - 3x + 3x^2 - x^3)^2$.	*16.*	$(\frac{1}{2}a + a^2b + \frac{1}{2}b)^2$.
13.	$(\frac{3}{4}x - 1\frac{1}{2}y + 1\frac{1}{4}z)^2$.	*17.*	$(m^2 + \frac{1}{2}m + \frac{1}{2})^2$.
14.	$(2x^3 + x - 7)^2$.	*18.*	$(5x + 3y^2 - .25z)^2$.

226. To find the cube of a binomial.

I. Required to find the value of $(x+y)^3$. Also of $(x-y)^3$.

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2\\ &\quad\ \underline{x + y}\\ &\quad\ x^3 + 2x^2y + xy^2\\ &\quad\ \underline{\qquad\ x^2y + 2xy^2 + y^3}\\ (x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

$$\begin{aligned}(x-y)^2 &= x^2 - 2xy + y^2\\ &\quad\ \underline{x - y}\\ &\quad\ x^3 - 2x^2y + xy^2\\ &\quad\ \underline{\qquad - x^2y + 2xy^2 - y^3}\\ (x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3\end{aligned}$$

II. Required to find the cube of $2x - 3y$.

$$\begin{aligned}(2x-3y)^2 &= 4x^2 - 12xy + 9y^2\\ &\quad\ \underline{2x - 3y}\\ &\quad\ 8x^3 - 24x^2y + 18xy^2\\ &\quad\ \underline{\qquad - 12x^2y + 36xy^2 - 27y^3}\\ (2x-3y)^3 &= 8x^3 - 36x^2y + 54xy^2 - 27y^3\end{aligned}$$

Rule.—*Write the cube of the first term; three times the square of the first term multiplied by the second term; three times the first term multiplied by the square of the second term; and the cube of the second term.*

EXERCISES.

227. ***Write the values of the following indicated cubes:***

1. $(a + b)^3$.
2. $(2x + y)^3$.
3. $(3m - 2n)^3$.
4. $(2a - 3b)^3$.
5. $(3a + 4b)^3$.
6. $(4a - 3y)^3$.
7. $(m - 2n)^3$.
8. $(a + 3b)^3$.
9. $(2a - b)^3$.
10. $(1 + x)^3$.
11. $(1 - x^2)^3$.
12. $(x - 2y)^3$.
13. $(a + 5)^3$.
14. $(\frac{1}{2}x - \frac{1}{2}y)^3$.
15. $(1 - mn)^3$.
16. $(2a + c)^3$.
17. $(\frac{1}{4}x + \frac{1}{2}y)^3$.
18. $(4x + y)^3$.
19. $(\frac{2}{3}a + b)^3$.
20. $(x - \frac{1}{3}y)^3$.
21. $(a^2 + b^2)^3$.
22. $(x^2 - 2y)^3$.
23. $(m - n^2)^3$.
24. $(3a^2 + b^2)^3$.
25. $(x^3 + 2x^2y)^3$.
26. $(x^2 + xy)^3$.
27. $(2x^2 - 3xy)^3$.
28. $(3ab - 2b^2)^3$.

228. To find the cube of a trinomial.

Required to find the cube of $a^2 - 2a + 5$.

$$\begin{aligned}(a^2 - 2a + 5)^3 &= \{(a^2 - 2a) + 5\}^3 \\ &= (a^2 - 2a)^3 + [3(a^2 - 2a)^2 \times 5] \\ &\qquad + [3(a^2 - 2a) \times 5^2] + 5^3 \\ &= (a^6 - 6a^5 + 12a^4 - 8a^3) + (15a^4 - 60a^3 + 60a^2) \\ &\qquad + (75a^2 - 150a) + 125 \\ &= a^6 - 6a^5 + 27a^4 - 68a^3 + 135a^2 - 150a + 125.\end{aligned}$$

III. *Double the root already found, for a trial divisor. Divide the remainder by the trial divisor, writing the quotient as the second figure of the root, and annexing it to the trial divisor. Multiply the complete divisor by this last root figure, subtract, and bring down the next period as before.*

IV. *So continue to the last period, after which, if additional places are required in the root, extend the operation by bringing down cipher periods.*

V. *Should the trial divisor at any time not be contained in the remainder, write a cipher in the root place, annex a cipher to the trial divisor, and bring down another period.*

NOTE.—The square root of a common fraction is found by extracting the square root of its numerator and of its denominator, and writing them as the numerator and denominator, respectively, of the root; or by reducing the common fraction to a decimal and proceeding in accordance with the rule.

EXERCISES.

249. ***Find the square roots of the following:***

1.	576.	*11.*	21025.	*21.*	$\frac{625}{1089}$.
2.	784.	*12.*	93025.	*22.*	$\frac{4225}{5041}$.
3.	1369.	*13.*	929296.	*23.*	$\frac{18769}{29929}$.
4.	2025.	*14.*	106929.	*24.*	$\frac{58081}{100489}$.
5.	4489.	*15.*	893025.	*25.*	18567481.
6.	4096.	*16.*	167281.	*26.*	37075921.
7.	8649.	*17.*	380689.	*27.*	1837.8369.
8.	9409.	*18.*	1.752976.	*28.*	18.533025.
9.	11449.	*19.*	4721929.	*29.*	.56370064.
10.	15129.	*20.*	2271049.	*30.*	975353.76.

II. Required to find the square root of 11 to 4 decimal places. Also the square root of $\frac{2}{3}$ to 5 decimal places.

11 | 3.3166+	$\frac{2}{3} = .6666+$	.66'66 | .81649+
9		64
63 | 200		161 | 266
189		161
661 | 1100		1626 | 10566
661		9756
6626 | 43900		16324 | 81066
39756		65296
66326 | 414400		163289 | 1577066
397956		1469601

Find the square roots to 4 decimal places:

31. 21.	*34.* 41.	*37.* $\frac{5}{8}$.	*40.* $\frac{1}{6}$.
32. .05.	*35.* 113.	*38.* $1\frac{1}{12}$.	*41.* $3\frac{3}{4}$.
33. .009.	*36.* 624.	*39.* $\frac{4}{5}$.	*42.* $7\frac{9}{13}$.

250. To find the cube root of polynomials.

I. Required to find the cube root of $x^3 + 3x^2y + 3xy^2 + y^3$.

Process.

$x^3 + 3x^2y + 3xy^2 + y^3 \mid x + y$

x^3

$3x^2 + 3xy + y^2 \mid 3x^2y + 3xy^2 + y^3$

$3x^2y + 3xy^2 + y^3$

Explanation.—For the sake of simplicity, a polynomial is chosen whose cube root is already known to be $x + y$. In cubing $x + y$, the first term of the result is x^3; hence, the first term of the cube root of the polynomial is found by taking the cube root of x^3. The second term of the power is $3x^2y$, which is 3 times the square of the first term of the

root multiplied by the second term ; hence, the second term root is found by dividing $3x^2y$ by 3 times the square of x. After taking away x^3, the cube of the first term of the root, the may be written, $(3x^2 + 3xy + y^2)y$. Hence, the operation is by adding $3xy$ and y^2 to $3x^2$ and multiplying the result by y.

II. Required to find the cube root of $x^6 - 3x^5 + 5x^3$

Process.

$$
\begin{array}{l|l}
 & x^6 - 3x^5 + 5x^3 - 3x - 1 \;\big|\; x^2 \\
 & x^6 \\
\hline
3x^4 - 3x^3 + x^2 & \quad - 3x^5 \\
 & \quad - 3x^5 + 3x^4 - x^3 \\
\hline
3x^4 - 6x^3 + 3x^2 & - 3x^4 + 6x^3 - 3x - \\
\quad - 3x^3 + 3x + 1 & \\
\hline
3x^4 - 6x^3 + 3x + 1 & - 3x^4 + 6x^3 - 3x -
\end{array}
$$

Explanation.—Take the cube root of x^6 for the first term of subtract its cube from the polynomial ; bring down $-3x^5$, the of the remainder.

Divide $-3x^5$ by 3 times the square of the first term of the is, by $3x^4$. This gives $-x$, the second term of the root.

To the divisor, $3x^4$, add 3 times the product of the first te root by the second term of the root ; add also the square of t term of the root.

Multiply the divisor thus completed by $-x$, the second te root, and subtract the product from the remainder.

Proceed in the same way to find the third term of the root.

EXERCISES.

251. ***Find the cube root of the following:***

1. $x^3 - 3x^2y + 3xy^2 - y^3$.
2. $m^3 + 3m^2n + 3mn^2 + n^3$.
3. $x^3 + 9x^2 + 27x + 27$.

4. $8a^3 + 36a^2 + 54a + 27.$

5. $27x^3 - 54x^2 + 36x - 8.$

6. $x^6 - 6x^4 + 12x^2 - 8.$

7. $27y^6 - 135y^4 + 225y^2 - 125.$

8. $a^3x^3 + 6a^2x^2 + 12ax + 8.$

9. $125x^6 - 75x^4 + 15x^2 - 1.$

10. $8x^6 - 60x^4y^3 + 150x^2y^6 - 125y^9.$

11. $125a^6b^3 - 225a^4b^2 + 135a^2b - 27.$

12. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$

13. $x^6 - 9x^5 + 33x^4 - 63x^3 + 66x^2 - 36x + 8.$

14. $8x^6 - 36ax^5 + 102a^2x^4 - 171a^3x^3 + 204a^4x^2 - 144a^5x + 64a^6.$

15. $8m^6 + 36m^5n + 42m^4n^2 - 9m^3n^3 - 21m^2n^4 + 9mn^5 - n^6.$

16. $27a^6 - 135a^5 + 171a^4 + 55a^3 - 114a^2 - 60a - 8.$

17. $216a^{12} - 432a^{10} - 36a^8 + 368a^6 + 18a^4 - 108a^2 - 27.$

18. $8x^6 - 36x^5y + 114x^4y^2 - 207x^3y^3 + 285x^2y^4 - 225xy^5 + 125y^6.$

19. $27a^6 - 108a^5 - 45a^4 + 440a^3 + 105a^2 - 588a - 343.$

20. $8x^6 + 48ax^5 + 60a^2x^4 - 80a^3x^3 - 90a^4x^2 + 108a^5x - 27a^6.$

252. To extract the cube root of numbers.

The cube of a number contains three times as many figures as the number itself, three times as many figures less one, or three times as many figures less two. Thus,

Numbers:	2,	3,	5,	12,	99,	.06,	3.5.
Cubes:	8,	27,	125,	1728,	970299,	.000216,	42.875.

Hence, *If the cube of any number be separated into periods of three figures each, beginning at the right, there will be as many periods as there are figures in the cube root.*

NOTE 1.—The left-hand period may contain one, two, or three figures.

NOTE 2.—Decimals are separated into periods by beginning at the decimal point. The right-hand decimal period must contain three figures, ciphers being annexed, if necessary.

253. Any number, as 85, may be cubed in the same manner as an algebraic binomial. Thus,

$$(85)^3 = (80 + 5)^3 = 80^3 + 3\,(80^2 \times 5) + 3\,(80 \times 5^2) + 5^3$$

$$= 80^3 + [(3 \times 80^2) + (3 \times 80 \times 5) + 5^2] \times 5$$

$$= 512000 + (19200 + 1200 + 25) \times 5 = 614125.$$

Hence, *The cube of any number composed of tens and units is equal to* $\textit{tens}^3 + (3\ \textit{times}\ \textit{tens}^2 + 3\ \textit{times}\ \textit{tens} \times \textit{units} + \textit{units}^2) \times \textit{units}$.

NOTE.—The same method is applicable when there are more than two figures in the number. Thus,

$$(125)^3 = (120 + 5)^3 = 120^3 + [(3 \times 120^2) + (3 \times 120 \times 5) + 5^2] \times 5.$$

EXERCISES.

254. ***By the foregoing method find the cube of:***

1. 12.	*5.* 45.	*9.* 81.	*13.* 105.
2. 15.	*6.* 57.	*10.* 28.	*14.* 208.
3. 21.	*7.* 63.	*11.* 93.	*15.* 195.
4. 34.	*8.* 49.	*12.* 86.	*16.* 306.

I. Required to find the cube root of 614125.

Process.

$$\begin{array}{lrl|l} & & & 614'125 & 85 \\ & & & 512 & \\ \hline 3\times 80^2 & = & 19200 & 102\,125 & \\ 3\times 80\times 5 & = & 1200 & & \\ 5^2 & = & 25 & & \\ & & 20425 & 102\,125 & \end{array}$$

Explanation.—Separating 614125 into periods of three figures each shows that the cube root contains two figures. Whatever the tens' figure of the root is, the significant figures of its cube are included in the left-hand period. The greatest cube in this period is 512 thousand, and its cube root is 8 tens or 80. Writing 8 as the tens of the root, and subtracting its cube, 512 thousand, from 614125, leaves 102125. This 102125 consists of (*3 times the tens*^{2} + *3 times the tens* × *the units* + *units*^{2}) × *units*. Since the second and third terms in the former of these two factors are small as compared with the first term, 102125 may be regarded as composed of *3 times the tens*^{2} × *units*. Hence, if 102125 be divided by *3 times the tens*^{2}, or 19200, the quotient will be the units' figure of the root, equal to 5. Knowing now the units' figure of the root, the second and third terms of the first factor of 102125 may be found. Adding the second term, 1200, and the third term, 25, to 19200, the first term, gives the first factor complete, equal to 20425. Multiplying this by the second factor, 5, gives 102125. Subtracting this product from the remainder, nothing remains, showing that 614125 is a perfect cube, of which 85 is the cube root.

Rule.—I. *Separate the given number into periods of three figures each, counting to the left and right from the units' place.*

II. *The first figure of the root will be the cube root of the greatest cube contained in the left-hand period. Subtract this cube from the left-hand period, and to the remainder annex the next period to form a dividend.*

III. *For a trial divisor take three times the square of the root figure, regarded as tens, and divide the dividend by it to obtain the second figure of the root.*

IV. *For a complete divisor add to the trial divisor*

three times the second figure of the root multiplied by the first figure considered as tens, and the square of the second figure of the root.

V. *From the dividend subtract the product of the complete divisor by the second figure of the root.*

VI. *Bring down the next period, if any remain. Treating the part of the root already found as if it were a single figure, proceed as before to find the next trial divisor, the next figure of the root, the complete divisor, and the next dividend.*

NOTE 1.—If the trial divisor is not contained in the dividend, annex a cipher to the root, two ciphers to the trial divisor, and another period to the dividend. Then divide the dividend by the trial divisor for the next figure of the root.

NOTE 2.—When the number whose cube root is to be found is not a perfect cube, decimal places may be obtained in the root by annexing cipher periods to each successive remainder.

EXERCISES.

255. ***Find the cube roots of the following numbers:***

1.	4913.	*11.*	743677416.
2.	68921.	*12.*	356400829.
3.	314432.	*13.*	1879080.904.
4.	912673.	*14.*	183056.925752.
5.	1860867.	*15.*	34.169931125.
6.	41063625.	*16.*	.564379103744.
7.	311665752.	*17.*	352643.999625.
8.	122763473.	*18.*	780.721126912.
9.	521660125.	*19.*	27189.441343.
10.	68417929.	*20.*	706157817625.

EXPONENTS.

256. Exponents may be **positive** or **negative, integral** or **fractional, numerical** or **literal.**

257. The exponents so far employed in this book have been positive whole numbers. Evidently the meaning of exponents that are not positive whole numbers should be such that the laws governing their use may be consistent with what has already been learned about exponents.

258. By the law of exponents in multiplication,

$$a^3 \times a^2 = a^{3+2} = a^5;$$

Hence,

$$a^{-3} \times a^2 = a^{-3+2} = a^{-1};$$

$$a^3 \times a^{-2} = a^{3-2} = a;$$

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{2}{2}};$$

$$a^{\frac{1}{2}} \times a^{-\frac{1}{2}} = a^{\frac{1}{2}-\frac{1}{2}} = a^{-\frac{1}{2}};$$

$$a^{\frac{1}{2}} \times a^{-\frac{1}{2}} = a^{\frac{1}{2}-\frac{1}{2}} = a^0 = 1 \text{ (see Art. 105)};$$

$$a^{\frac{m}{n}} \times a^{-\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}};$$

$$m^{\frac{1}{2}}n \times m^{-\frac{1}{2}}n = m^0 n = n.$$

EXERCISES.

259. ***Multiply:***

1. $a^{\frac{1}{2}}$ by $a^{\frac{1}{2}}$.
2. $x^{\frac{1}{2}}y$ by $xy^{\frac{1}{2}}$.
3. $x^{\frac{1}{2}}y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}y^{\frac{1}{2}}$.
4. $m^{\frac{1}{2}}$ by $m^{\frac{1}{2}}n^{\frac{1}{2}}$.

5. x^2 by x^{-1}.
6. a^3 by $a^{-\frac{3}{4}}$.
7. a^2 by $a^{-\frac{1}{4}}$.
8. $3a^{-1}$ by $6a^3b^2$.
9. $2x^{\frac{3}{4}}$ by $3x^{\frac{1}{4}}$.
10. $2x^{-1}$ by $2x^{-2}$.
11. $m^{\frac{3}{4}}n^{\frac{3}{4}}$ by $m^{-\frac{3}{4}}$.
12. a^2b^{-2} by $a^{-1}b^{-\frac{1}{4}}$.
13. $x^{\frac{1}{4}}y$ by $x^{\frac{3}{4}}y^{-1}$.
14. a^n by $a^{2n}b^m$.
15. $x^{-n}y \times x^{\frac{n}{3}}y^m$.
16. $a^{\frac{n}{3}} \times a^{\frac{n}{2}}$.
17. $a^{\frac{1}{n}}b \times a^{\frac{2}{n}}b^{\frac{1}{4}}$.
18. $a^{-4} \times a^2b^{-3}$.
19. $m^{n+p} \times m^{2n-p}$.
20. $x^{a-2b} \times x^{b-a}$.
21. $x^{-\frac{a}{3}} \times x^{-\frac{2a}{3}}$.
22. $x^{-1}y^{-\frac{3}{4}} \times xy^{\frac{3}{4}}$.

I. Required to multiply $a^{\frac{1}{2}} + a^{\frac{1}{4}}x^{\frac{1}{4}} + x^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - a^{\frac{1}{4}}x^{\frac{1}{4}} + x^{\frac{1}{2}}$.

$$
\begin{array}{l}
a^{\frac{1}{2}} + a^{\frac{1}{4}}x^{\frac{1}{4}} + x^{\frac{1}{2}} \\
a^{\frac{1}{2}} - a^{\frac{1}{4}}x^{\frac{1}{4}} + x^{\frac{1}{2}} \\
\hline
a \;+ x^{\frac{1}{4}}x^{\frac{1}{4}} + a^{\frac{1}{4}}x^{\frac{1}{2}} \\
\quad - a^{\frac{1}{4}}x^{\frac{1}{4}} - a^{\frac{1}{4}}x^{\frac{1}{2}} - a^{\frac{1}{4}}x \\
\qquad\qquad + a^{\frac{1}{4}}x^{\frac{1}{2}} + a^{\frac{1}{4}}x + x^{\frac{1}{4}} \\
\hline
a \qquad\qquad + a^{\frac{1}{4}}x^{\frac{1}{2}} \qquad\quad + x^{\frac{1}{4}}
\end{array}
$$

II. Required to multiply $x^{\frac{4}{3}} - xy^{\frac{1}{3}} + x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} + y$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

$$
\begin{array}{l}
x^{\frac{4}{3}} - xy^{\frac{1}{3}} + x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} + y \\
x^{\frac{1}{3}} + y^{\frac{1}{3}} \\
\hline
x^{\frac{5}{3}} - x^{\frac{4}{3}}y^{\frac{1}{3}} + xy^{\frac{1}{3}} - x^{\frac{2}{3}}y^{\frac{2}{3}} + x^{\frac{1}{3}}y \\
\quad + x^{\frac{4}{3}}y^{\frac{1}{3}} - xy^{\frac{1}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}y + y^{\frac{2}{3}} \\
\hline
x^{\frac{5}{3}} \qquad\qquad\qquad\qquad\qquad\qquad + y^{\frac{5}{3}}
\end{array}
$$

EXERCISES.

260. ***Multiply:***

1. $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$. 2. $x^{\frac{1}{3}} - y^{\frac{1}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

3. $m^{\frac{1}{2}} - n^{\frac{1}{2}}$ by $m^{\frac{1}{2}} + n^{\frac{1}{2}}$.

4. $x^{-\frac{1}{3}} + y^{-\frac{1}{3}}$ by $x^{-\frac{1}{3}} - y^{-\frac{1}{3}}$.

5. $a^{-\frac{1}{2}} - b^{-\frac{1}{2}}$ by $a^{-\frac{1}{2}} + b^{-\frac{1}{2}}$.

6. $x^{-\frac{1}{2}} + y^{\frac{1}{2}}$ by $x^{-\frac{1}{2}} + y^{\frac{1}{2}}$.

7. $x^{-3} + y^{2}$ by $x^{2} - y^{-3}$.

8. $p^{\frac{1}{4}} - q^{-\frac{1}{4}}$ by $p^{\frac{1}{4}} + q^{\frac{1}{4}}$.

9. $x^{-2} - x^{-1}y^{-1} + y^{-2}$ by $x^{-2} + x^{-1}y^{-1} + y^{-2}$.

10. $a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$.

11. $x^{-\frac{1}{2}} - x^{-1}y^{\frac{1}{2}} + x^{-\frac{1}{2}}y - y^{\frac{1}{2}}$ by $x^{-1} + x^{-\frac{1}{2}}y^{\frac{1}{2}} - y$.

12. $x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 3x^{\frac{1}{4}} + 2x^{\frac{1}{4}} + 1$ by $x^{\frac{1}{2}} - 2x^{\frac{1}{4}} + 1$.

13. $x^{-2} + 4x^{-\frac{1}{2}} + 13x^{-1} + 42x^{-\frac{1}{2}} + 28$ by $x^{-\frac{1}{2}} - 4x^{-1} + 3x^{-\frac{1}{2}} - 2$.

14. $x^{-\frac{1}{2}} + 62x^{-1} + 11x^{-\frac{3}{2}} + 3x^{-\frac{1}{2}} + 2$ by $5x^{-\frac{1}{2}} - 7x^{-\frac{1}{2}} + 1$.

15. $.2x^{2m} - .8x^{n}y^{-r} + 1.6y^{-2r}$ by $.1x^{2m} + .4x^{n}y^{-r} + .8y^{-2r}$.

16. $5x^{-4} + 7x^{-3} + 44x^{-2} + 7x^{-1} + 1$ by $5x^{-2} - 7x^{-1} + 1$.

17. $\frac{1}{3}x^{-1} - .2x^{-\frac{1}{2}}y^{\frac{1}{2}} + .06y^{\frac{3}{2}}$ by $.5x^{-1} + .3x^{-\frac{1}{2}}y^{\frac{1}{2}} + .09y^{\frac{2}{3}}$.

18. $\frac{5}{6}m - \frac{1}{3}m^{\frac{3}{2}} + \frac{1}{20}m^{\frac{1}{2}} - \frac{1}{80}$ by $5m^{\frac{3}{2}} + 2m^{\frac{1}{2}} + \frac{1}{2}$.

19. $\frac{2}{3}x^{-\frac{3}{2}} - \frac{2}{5}x^{-\frac{1}{2}}y^{-1} + \frac{2}{5}y^{-2}$ by $\frac{1}{3}x^{-\frac{3}{2}} + \frac{1}{5}x^{-\frac{1}{2}}y^{-1} + \frac{1}{5}y^{-2}$.

20. $42x^{\frac{1}{4}} + 105x + 23x^{\frac{3}{4}} + 5x^{\frac{1}{4}} + 1$ by $2x^{\frac{3}{4}} - 5x^{\frac{1}{4}} + 1$.

261. By the laws of exponents in division,

$$a^5 \div a^2 = a^{5-2} = a^3.$$

Hence,
$$a^5 \div a^{-2} = a^{5-(-2)} = a^7;$$
$$a^{-5} \div a^{-2} = a^{-5-(-2)} = a^{-3}.$$
$$a^{\frac{1}{2}} \div a^{\frac{1}{2}} = a^{\frac{1}{2}-\frac{1}{2}} = a^{\frac{1}{2}};$$
$$a^{-\frac{1}{2}} \div a^{-\frac{1}{2}} = a^{-\frac{1}{2}-(-\frac{1}{2})} = a^{-\frac{1}{2}}.$$
$$a^m \div a^n = a^{m-n};$$
$$a^m \div a^{-n} = a^{m-(-n)} = a^{m+n}.$$
$$a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}};$$
$$a^{\frac{m}{n}} \div a^{-\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}}.$$

EXERCISES.

262. *Divide:*

1. $12a^3$ by $4a^5$.
2. $8x^{-2}$ by $2x^2$.
3. $6x^3$ by $3x^{-2}$.
4. $a^{\frac{1}{2}}b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}b$.
5. $x^{\frac{1}{2}}y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}y^{\frac{1}{2}}$.
6. $x^{-1}y^2$ by $x^{-2}y^{-1}$.
7. $15a^{\frac{1}{2}}b^{\frac{1}{2}}$ by $3a^{-\frac{1}{2}}b^{-\frac{1}{2}}$.
8. $8x^{2m}y^{-2}$ by $4x^{4m}y^{-3}$.
9. $a^{\frac{1}{2}}b^2c^{-1}$ by bc^2x^{-1}.
10. $8b^{\frac{1}{2}}cd$ by $b^{-1}c^2x^{-1}$.
11. $a^{\frac{1}{2}}b^{\frac{3}{2}}c^{-1}$ by abc.
12. $x^{-2}y^{\frac{3}{2}}$ by $x^{-3}y^{-\frac{1}{2}}z$.
13. $10mn^{-\frac{3}{2}}$ by $2m^{-1}n^{\frac{1}{2}}x^2$.
14. $3x^ny^{-\frac{1}{2}}z^2$ by $6x^{n-1}y^{-\frac{3}{2}}z^3$.
15. $4m^{n+1}n^2x^{-1}$ by $6m^2n^2x^{-1}$.
16. $12a^{-2}x^{\frac{3}{2}} - 18abx^{-\frac{1}{2}}$ by $6ab^2x$.

17. $16x^{n}y^{2} + 20x^{-1}y^{\frac{3}{4}}$ by $24xyz$.

18. $24a^{-1}b^{n}c^{2} - 30a^{2}b^{\frac{1}{2}}x^{-2}$ by $36abc^{2}$.

19. $6m^{\frac{1}{2}}n^{\frac{3}{4}} - 8m^{2}n^{\frac{3}{4}}x$ by $4a^{-1}m^{2}n^{-\frac{1}{2}}$.

20. $10x^{\frac{3}{4}}yz^{\frac{1}{2}} + 15a^{-1}x^{2}y^{\frac{3}{4}}$ by $25a^{-1}yz^{-n}$.

21. $16a^{-\frac{1}{2}}b^{-1}c^{-\frac{1}{2}} - 20a^{\frac{1}{2}}b^{n}c^{-1}$ by $4abc$.

22. $\frac{3}{5}x^{n}y^{r}z^{-1} + \frac{3}{5}x^{-1}y^{-2}z^{-\frac{1}{2}}$ by $\frac{1}{5}x^{-1}yz^{-2}$.

23. $\frac{5}{8}x^{n}y^{-1}z^{-2} - \frac{1}{8}x^{-\frac{1}{2}}yz^{\frac{1}{2}}$ by $\frac{1}{4}x^{-\frac{1}{2}}y^{-1}z^{-\frac{3}{2}}$.

I. Required to divide $m^{\frac{3}{2}} - n^{\frac{3}{2}}$ by $m^{\frac{1}{2}} - n^{\frac{1}{2}}$.

$$
\begin{array}{l|l}
m^{\frac{3}{2}} - n^{\frac{3}{2}} & m^{\frac{1}{2}} - n^{\frac{1}{2}} \\
\underline{m^{\frac{3}{2}} - mn^{\frac{1}{2}}} & m + m^{\frac{1}{2}}n^{\frac{1}{2}} + n^{\frac{1}{2}} \\
\quad mn^{\frac{1}{2}} - n^{\frac{3}{2}} & \\
\quad \underline{mn^{\frac{1}{2}} - m^{\frac{1}{2}}n^{\frac{1}{2}}} & \\
\quad\quad m^{\frac{1}{2}}n^{\frac{1}{2}} - n^{\frac{3}{2}} & \\
\quad\quad \underline{m^{\frac{1}{2}}n^{\frac{1}{2}} - n^{\frac{3}{2}}} &
\end{array}
$$

II. Required to divide $x^{\frac{3}{2}} + x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$ by $x^{\frac{1}{2}} - x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-\frac{1}{2}}$.

$$
\begin{array}{l|l}
x^{\frac{3}{2}} + x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1} & x^{\frac{1}{2}} - x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-\frac{1}{2}} \\
\underline{x^{\frac{3}{2}} - x^{\frac{1}{2}}y^{-\frac{1}{2}} + x^{\frac{1}{2}}y^{-\frac{1}{2}}} & x^{\frac{1}{2}} + x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-\frac{1}{2}} \\
\quad x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1} & \\
\quad \underline{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{\frac{1}{2}}y^{-\frac{1}{2}} + x^{\frac{1}{2}}y^{-\frac{3}{2}}} & \\
\quad\quad x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{\frac{1}{2}}y^{-\frac{3}{2}} + y^{-1} & \\
\quad\quad \underline{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{\frac{1}{2}}y^{-\frac{3}{2}} + y^{-1}} &
\end{array}
$$

EXERCISES.

263. ***Divide:***

1. $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.
2. $x^{\frac{3}{2}} - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.
3. $x - y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.
4. $m^{\frac{3}{2}} + n^{\frac{3}{2}}$ by $m^{\frac{1}{2}} + n^{\frac{1}{2}}$.
5. $p^{\frac{3}{2}} - q^{\frac{3}{2}}$ by $p^{\frac{1}{2}} - q^{\frac{1}{2}}$.
6. $x^{-\frac{1}{2}} - y^{\frac{1}{2}}$ by $x^{-\frac{1}{2}} - y^{\frac{1}{2}}$.
7. $a^{\frac{1}{2}} - x^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - x^{\frac{1}{2}}$.
8. $x^{\frac{1}{2}} + 8y^{-2}$ by $x^{\frac{1}{2}} + 2y^{-\frac{3}{2}}$.
9. $x^{n} + 8x^{\frac{n}{2}} + 15$ by $x^{\frac{n}{2}} + 5$.
10. $x^{-3} - 17x^{-3} + 7x^{-2} - 9$ by $x^{-2} + 5x^{-1} + 3$.
11. $a^{-1} - 82a^{-\frac{1}{2}} - 3$ by $a^{-\frac{1}{2}} + 3$.
12. $m^{2n} + m^{n}x^{p} + x^{2p}$ by $m^{n} - m^{\frac{n}{2}}x^{\frac{p}{2}} + x^{p}$.
13. $4x^{3} + 11x^{2} + 21x + 6x^{\frac{1}{2}} - 10$ by $2x - 3x^{\frac{1}{2}} + 5$.
14. $2x^{2} - 9x^{\frac{3}{2}} - 8x - 1$ by $x + 3x^{\frac{1}{2}} + 3x^{\frac{1}{2}} + 1$.
15. $x - 17x^{\frac{3}{4}} + 13x^{\frac{1}{2}} + 15x^{\frac{1}{4}} - 12$ by $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 4$.
16. $x^{\frac{7}{4}} - 109x - 12x^{\frac{3}{4}} - 56$ by $x - 4x^{\frac{3}{4}} + 3x^{\frac{1}{2}} - 2$.
17. $2a^{-1} - 10a^{-\frac{3}{4}} - 38a^{-\frac{3}{4}} - 98$ by $a^{-\frac{3}{4}} + 4a^{-\frac{1}{2}} + 7$.
18. $8x^{-2} - 95x^{-1} + 1$ by $2x^{-\frac{3}{4}} - 5x^{-\frac{1}{2}} + 1$.
19. $x^{2} + x^{\frac{5}{4}} - 11x^{\frac{1}{2}} + 9$ by $x^{\frac{3}{4}} - 2x^{\frac{1}{2}} + 1$.
20. $x^{\frac{7}{4}} + 7x^{\frac{1}{4}} - 21x^{-\frac{1}{4}} - 21x^{-\frac{5}{4}} - 6x^{-\frac{7}{4}}$ by $x^{\frac{3}{4}} + 3x^{\frac{1}{4}} + 3x^{-\frac{1}{4}} + x^{-\frac{3}{4}}$.

264. In raising a monomial to any required power, it has been shown that the exponents of the factors in the result are found by multiplying the exponents of the factors in the monomial by the exponent of the power. This law is true whatever the form of the exponent. Thus,

$$(a^2x^3y^4)^2 = a^{2\times2}x^{3\times2}y^{4\times2} = a^4x^6y^8.$$

$$(a^nx^{2p}y^{3q})^3 = a^{n\times3}x^{2p\times3}y^{3q\times3} = a^{3n}x^{6p}y^{9q}.$$

$$(a^{\frac{1}{2}}x^{-2}y^{-\frac{1}{2}})^2 = a^{\frac{1}{2}\times2}x^{-2\times2}y^{-\frac{1}{2}\times5} = ax^{-4}y^{-3}.$$

$$(a^{-\frac{1}{2}}x^{-n}y^{-\frac{2}{3}})^{\frac{2}{3}} = a^{-\frac{1}{2}\times\frac{2}{3}}x^{-n\times\frac{2}{3}}y^{-\frac{2}{3}\times\frac{2}{3}} = a^{-\frac{1}{3}}x^{-\frac{2n}{3}}y^{-1}.$$

EXERCISES.

265. ***Find the values of the following:***

1. $(x^2y^{-3}z)^3$.
2. $(2a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-\frac{1}{2}})^4$.
3. $(3m^nx^{-\frac{1}{2}}y^{-1})^2$.
4. $(5b^{-2}c^{n-1}d^{n+1})^2$.
5. $(10a^{-\frac{2}{3}}b^{\frac{1}{3}}c^{-\frac{1}{3}})^3$.
6. $(2^2x^{-2}y^{-\frac{2}{3}}z^n)^3$.
7. $(3^{\frac{1}{3}}a^{-\frac{2}{3}}b^{-1}x^{\frac{1}{3}})^3$.
8. $(8a^{\frac{1}{2}}b^{-3}c^{\frac{3}{4}})^{\frac{2}{3}}$.
9. $(4^{-\frac{1}{2}}x^{\frac{1}{4}}y^{-3m}z^n)^{-\frac{1}{2}}$.
10. $(5^{\frac{1}{n}}p^{\frac{m}{n}}q^{-1}r^{-\frac{1}{2}})^n$.
11. $(\frac{3}{4}a^{\frac{2}{3}}m^{-\frac{1}{2}}y^{-\frac{1}{2}})^2$.
12. $(7^{-\frac{2}{3}}a^{\frac{2}{5}}x^{-\frac{2}{10}}y^{-1})^{-5}$.
13. $(3^{-\frac{1}{p}}x^{-1}y^{\frac{2}{3}}z^{-\frac{n}{p}})^{-p}$.
14. $(4^{\frac{n}{m}}a^{2n}b^{\frac{3n}{m}}c^{-\frac{1}{2}})^{\frac{2m}{n}}$.
15. $(5^{\frac{2}{3}}x^{\frac{1}{2}}y^{-\frac{1}{3}}z)^{\frac{3}{2}}$.
16. $(10^{\frac{2}{3}}x^{-\frac{1}{6}}y^{\frac{2}{3}}z^n)^6$.
17. $(12^{\frac{2}{3}}a^{-\frac{1}{6}}b^{-1}m^n)^3$.
18. $(16^{\frac{m}{n}}x^{\frac{1}{n}}y^{-1}z^{\frac{2}{n}})^{\frac{n}{m}}$.

266. It has been shown that in finding any required root of a monomial, the exponents of the factors in the root are found by dividing the exponents of the factors in the monomial by the index of the root. This law is true whatever the form of the index. Thus,

$$\sqrt{a} = a^{\frac{1}{2}}; \qquad \sqrt[3]{a} = a^{\frac{1}{3}};$$

$$\sqrt[3]{a^2} = a^{\frac{2}{3}}; \qquad \sqrt[n]{a^m} = a^{\frac{m}{n}};$$

$$3\sqrt[3]{a^2} \times 2\sqrt[n]{x^m} = 3a^{\frac{2}{3}} \times 2x^{\frac{1}{m}} = 6a^{\frac{2}{3}}x^{\frac{1}{m}}.$$

When, however, the index of the root is negative or fractional, the operation is indicated by means of an exponent.

Thus, $\sqrt[-2]{a}$ is written $a^{-\frac{1}{2}}$; $\sqrt[\frac{1}{2}]{a}$ is written $a^{\frac{2}{1}}$;

$\sqrt[-\frac{1}{2}]{a}$ " " $a^{-\frac{2}{1}}$; $\sqrt[\frac{m}{n}]{a^m}$ " " a^n;

$\sqrt[-2]{a^2b^3}$ " " $a^{-1}b^{-\frac{3}{2}}$; $\sqrt[-\frac{1}{2}]{a^{\frac{1}{2}}b^{-3}}$ " " $a^{-1}b^2$.

EXERCISES.

267. *Express with exponents:*

1. $\sqrt{ax}$.
2. $\sqrt[3]{m^2}$.
3. $\sqrt[3]{mn^2}$.
4. $\sqrt[4]{a^2x^3}$.
5. $\sqrt[\frac{1}{2}]{a^2x}$.
6. $\sqrt[-3]{a^{-1}b}$.
7. $\sqrt[n]{x^{m}y^{n}}$.
8. $\sqrt[n]{a^{\frac{m}{2}}b^{m}}$.
9. $\sqrt{a\,(x-y)}$.
10. $\sqrt[\frac{1}{2}]{a\,(x+y)^{\frac{1}{4}}}$.
11. $\sqrt[-\frac{1}{2}]{a^3\,(m-n)^6}$.
12. $\sqrt[\frac{m}{n}]{a^m b\,(x^2+y)^{\frac{m}{n}}}$.
13. $\sqrt{a} \times \sqrt[3]{b} \times \sqrt[4]{x^3y^2}$.
14. $\sqrt[n]{a^m} \times \sqrt[p]{b^2} \times \sqrt[q]{x^{\frac{1}{2}}}$.

15. $5\sqrt[3]{a^2} \times \sqrt[3]{mx^{-3}} \times \sqrt[4]{ab^{\frac{1}{2}}}$.

16. $3\sqrt[3]{(a-b)^2} \times 5\sqrt[6]{(a-b)^4}$.

268. ***Express with radical signs:***

1. $a^{\frac{1}{2}}$.	5. $a^{\frac{1}{2}}b^{\frac{1}{3}}$.	9. $3a^{\frac{2}{3}}b^{\frac{1}{4}}$.
2. $x^{\frac{1}{3}}y^{\frac{1}{4}}$.	6. $a^{\frac{2}{3}}b^{\frac{1}{4}}c^{\frac{1}{5}}$.	10. $5^{\frac{1}{2}}a^{\frac{1}{3}}b^{\frac{2}{3}}$.
3. $a^{\frac{1}{2}}x^{\frac{2}{3}}$.	7. $2ab^{\frac{1}{2}}$.	11. $16^{\frac{1}{4}}a^{\frac{2}{3}}b^{\frac{1}{2}}$.
4. $m^{\frac{2}{3}}n^{\frac{1}{4}}$.	8. $5a^{\frac{1}{2}}x$.	12. $8^{\frac{1}{3}}x^{\frac{1}{2}}y^{\frac{2}{3}}z^{\frac{1}{4}}$.

269. When a positive whole number is used as the exponent of a quantity, it indicates that the quantity is taken as a factor as many times as there are units in the exponent. But when the exponent is negative or fractional, it can have no such meaning. It is necessary, therefore, to examine these latter cases.

270. To find the meaning of integral negative exponents.

I. Required the meaning of a^{-3}. Also of $x^{-n}y^{-2}$.

$$a^{-3} = \frac{a^{-3}}{1}; \qquad \frac{a^{-3}}{1} \times a^3 = \frac{1}{a^3}.$$

$$x^{-n}y^{-2} = \frac{x^{-n}y^{-2}}{1}; \qquad \frac{x^{-n}y^{-2}}{1} \times x^n y^2 = \frac{1}{x^n y^2}.$$

II. Required the meaning of $\frac{x^{-2}}{y^{-3}}$. Also of $\frac{a^{-2}-b^{-2}}{a^{-1}+b^{-1}}$.

$$\frac{x^{-2}}{y^{-3}} = \frac{x^{-2}}{y^{-3}} \times x^2y^3 = \frac{y^3}{x^2}.$$

$$\frac{a^{-2}-b^{-2}}{a^{-1}+b^{-1}} = \frac{(a^{-2}-b^{-2}) \times a^2b^2}{(a^{-1}+b^{-1}) \times a^2b^2} = \frac{b^2-a^2}{ab^2+a^2b} = \frac{b-a}{ab}.$$

Hence, I. *Any quantity affected with an integral negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.*

II. *Any quantity affected with negative exponents may be cleared of them without changing its value.*

NOTE.—The reciprocal of a quantity is equal to 1 divided by that quantity.

Thus, the reciprocals of 2, $\frac{3}{4}$, a, $\frac{x}{y}$, and $\frac{mx}{ay}$ are respectively $\frac{1}{2}$, $\frac{4}{3}$, $\frac{1}{a}$, $\frac{y}{x}$, and $\frac{ay}{mx}$.

EXERCISES.

271. ***Clear the following of negative exponents:***

1. $\dfrac{a^{-2}}{x^{-1}}$.
2. $\dfrac{m^{-2}n^{-3}}{a^{-1}x^{-2}}$.
3. $\dfrac{5x^{-4}y^{-m}}{7a^{-3}b^{-2}}$.
4. $\dfrac{a^{-1}}{xy^{-1}}$.
5. $\dfrac{am^{-1}}{a+b^{-1}}$.
6. $\dfrac{3a^{2}b^{-1}}{3^{-1}a^{-2}b^{-2}}$.
7. $\dfrac{3a^{-2}b^{-3}x}{5^{-1}x^{-2}y^{-1}}$.
8. $\dfrac{(a+1)^{-2}}{(a+1)^{-1}}$.
9. $\dfrac{1}{a^{-1}+b^{-2}}$.
10. $\dfrac{a-2}{(a-2)^{-1}}$.
11. $\dfrac{x^{-2}+y^{-2}}{y^{-1}-x^{-1}}$.
12. $\dfrac{a^{-1}+b^{-2}}{a^{-2}+b^{-1}}$.
13. $\dfrac{x^{-3}-1}{x^{-1}-1}$.
14. $\dfrac{a+b}{a^{-1}+b^{-1}}$.
15. $\dfrac{a^{-1}+b}{b^{-1}+a}$.
16. $\dfrac{x-y}{y^{-1}-x^{-1}}$.
17. $\dfrac{a^{-m}-b^{-m}}{a^{-2m}-b^{-2m}}$.
18. $\dfrac{x^{-2}-y^{-2}}{x^{-4}-y^{-4}}$.
19. $\dfrac{5a\,(x-1)^{-1}-3\,(x-1)^{-1}}{(x-1)^{-2}}$.
20. $\dfrac{x^{-2}\,(x^{-1}-y^{-1})^{-1}-y^{-2}\,(x^{-1}-y^{-1})^{-1}}{x^{-1}\,(x^{-1}-y^{-1})^{-2}+y^{-1}\,(x^{-1}-y^{-1})^{-2}}$.

272. To find the meaning of fractional exponents.

I. Required the meaning of $a^{\frac{1}{2}}$. Also of $a^{\frac{2}{2}}$.

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a.$$

That is, $a^{\frac{1}{2}}$ taken twice as a factor gives a.

But the square root of a is also a quantity that when taken twice as a factor gives a.

Hence, $a^{\frac{1}{2}}$ is the square root of a; written also $\sqrt{a}$.

Again, $a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^2$.

That is, $a^{\frac{2}{3}}$ taken three times as a factor gives a^2.

But the cube root of a^2 is also a quantity that when taken three times as a factor will produce a^2.

Hence, $a^{\frac{2}{3}}$ is the cube root of a^2; written also $\sqrt[3]{a^2}$.

II. Required the meaning of $a^{-\frac{1}{2}}$. Also of $a^{-\frac{2}{4}}$.

$$a^{-\frac{1}{2}} \times a^{-\frac{1}{2}} = a^{-1} = \frac{1}{a}.$$

That is, $a^{-\frac{1}{2}}$ taken twice as a factor gives a^{-1}, or $\frac{1}{a}$.

But the square root of a^{-1}, or of $\frac{1}{a}$ is also a quantity that when taken twice as a factor gives $\frac{1}{a}$.

Hence, $a^{-\frac{1}{2}}$ is the square root of $\frac{1}{a}$; written also $\sqrt{\frac{1}{a}}$.

Again, $a^{-\frac{2}{4}} \times a^{-\frac{2}{4}} \times a^{-\frac{2}{4}} \times a^{-\frac{2}{4}} = a^{-2} = \frac{1}{a^2}$.

That is, $a^{-\frac{2}{4}}$ taken four times as a factor gives a^{-2}, or $\frac{1}{a^2}$.

But the fourth root of a^{-3}, or of $\frac{1}{a^3}$ is also a quantity that when taken four times as a factor gives a^{-3}, or $\frac{1}{a^3}$.

Hence, $a^{-\frac{3}{4}}$ is the fourth root of $\frac{1}{a^3}$; written also $\sqrt[4]{\frac{1}{a^3}}$.

A positive fractional exponent denotes a root of a power. The denominator of the exponent indicates the root, and the numerator indicates the power.

A negative fractional exponent denotes the reciprocal of what is denoted by an equal positive exponent.

EXERCISES.

273. ***Express the following without fractional or negative exponents:***

1. $x^{\frac{1}{3}}$.
2. $3a^{\frac{1}{2}}$.
3. $5a^{\frac{1}{2}}x^{\frac{1}{3}}$.
4. $9^{\frac{1}{2}}m^{\frac{1}{3}}n^{\frac{1}{4}}$.
5. $6a^{-\frac{2}{3}}x^{\frac{1}{2}}$.
6. $11m^{\frac{1}{3}}n^{-\frac{1}{4}}x^{\frac{1}{5}}$.
7. $5a^{\frac{1}{2}}x^{-\frac{1}{3}}y^{\frac{1}{4}}$.
8. $7x^{-\frac{1}{2}}y^{-\frac{1}{3}}z^{\frac{2}{3}}$.
9. $\frac{3a^{-\frac{1}{2}}b^{-\frac{1}{3}}}{4x^{-\frac{1}{2}}y^{-\frac{1}{3}}}$.
10. $\frac{2m^{\frac{1}{2}}n^{\frac{1}{3}}}{3^{-1}m^{-\frac{1}{2}}n^{-\frac{1}{3}}}$.
11. $\frac{4^{-\frac{1}{2}}a^{-\frac{1}{2}}b^{-\frac{1}{3}}}{9^{-\frac{1}{2}}a^{\frac{1}{2}}x^{-\frac{1}{3}}}$.
12. $\frac{25^{-\frac{1}{2}}a^{\frac{1}{2}}x^{-\frac{1}{3}}}{16^{-\frac{1}{2}}m^{\frac{1}{2}}n^{-\frac{1}{3}}x^{-\frac{2}{3}}}$.
13. $\frac{a^{-\frac{1}{2}}b^{-\frac{1}{3}}}{x^{-\frac{1}{2}}y^{-\frac{1}{3}}}$.
14. $\frac{a^{\frac{1}{2}}b^{-\frac{1}{3}}}{x^{-\frac{1}{2}}y^{\frac{1}{3}}}$.
15. $9\,(a+b)^{\frac{1}{2}}$.
16. $3\,(a-b)^{\frac{2}{3}}$.
17. $5a\,(x+3)^{\frac{2}{3}}$.
18. $17x^{\frac{1}{2}}\,(a-2)^{\frac{2}{3}}$.
19. $\frac{3a^{-\frac{1}{2}}\,(x+1)^{-\frac{1}{3}}}{4m^{-\frac{2}{3}}\,(x-1)^{-\frac{1}{3}}}$.
20. $13a^{\frac{1}{2}}\,(x^2+y)^{-\frac{2}{3}}$.

274. The following principle is useful in finding any required root of the product of several factors.

Any root of the product of two or more factors is equal to the product of the same roots of the factors.

Thus, $\sqrt{9 \times 16} = \sqrt{9} \times \sqrt{16} = 3 \times 4 = 12.$

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b};$$

and $$\sqrt[3]{xyz} = \sqrt[3]{x} \times \sqrt[3]{y} \times \sqrt[3]{z}.$$

EXERCISES.

275. ***Find the values of the following:***

1. $\sqrt{9 \times 4}$.	*9.* $\sqrt[3]{8 \times 27 \times 64}$.
2. $\sqrt{4 \times 16}$.	*10.* $\sqrt[3]{125 \times 216 \times 1000}$.
3. $\sqrt{9 \times 25}$.	*11.* $(121 \times 144 \times 49)^{\frac{1}{2}}$.
4. $\sqrt{36 \times 49}$.	*12.* $(64a^3 \times 343x^6)^{\frac{1}{3}}$.
5. $\sqrt{81 \times 64}$.	*13.* $(8 \times 27 \times 125)^{\frac{2}{3}}$.
6. $\sqrt{4 \times 9 \times 16}$.	*14.* $(16 \times 81 \times 256)^{\frac{3}{4}}$.
7. $\sqrt{16 \times 25 \times 64}$.	*15.* $(9 \times 64 \times 25)^{\frac{3}{2}}$.
8. $\sqrt{8^2 \times 9^2 \times 16}$.	*16.* $\sqrt[4]{16 \times 81 \times 625}$.

In the same way any required root of two or more factors that are not themselves perfect powers of the degree indicated may often be found.

Thus, $\sqrt{3 \times 75} = \sqrt{3 \times 3 \times 25} = \sqrt{3^2 \times 5^2} = 15.$

Also, $\sqrt{200 \times 128} = \sqrt{10^2 \times 2^2 \times 8^2} = 10 \times 2 \times 8 = 160.$

17. $\sqrt{300 \times 75 \times 121}$. *18.* $\sqrt[3]{192 \times 225 \times 40}$.

RADICALS.

276. A **radical quantity** or **radical** is a root of a quantity indicated by a radical sign or by a fractional exponent.

Thus, $\sqrt{a}$, $a^{\frac{1}{2}}$, $\sqrt[3]{x} + y$, $(x + y)^{\frac{1}{2}}$, are radical quantities.

277. The **co-efficient** of a radical is a quantity placed before the radical to show how many times it is to be taken. When no co-efficient is written, 1 is understood.

Thus, of the radicals $\sqrt{x}$, $2\sqrt[3]{a}$, $(a + b)\sqrt{x - y}$, the co-efficients are respectively 1, 2, and $(a + b)$.

278. The **degree** of a radical is denoted by the index of the radical, or by the denominator of the fractional exponent.

Thus, $\sqrt{ax}$, $(a + b)^{\frac{1}{2}}$, $2\sqrt{x - y}$, are of the *second* degree.

$2\sqrt[3]{a}$, $\sqrt[3]{a - b}$, $(mn)^{\frac{1}{3}}$, are of the *third* degree.

279. Similar radicals are those that express the same root of the same quantities.

Thus, $3\sqrt{ax}$, $-5\sqrt{ax}$, and $(a + b)(ax)^{\frac{1}{2}}$, are similar radicals.

REDUCTION OF RADICALS.

A radical is in its simplest form when it is not fractional and contains no perfect power of a degree denoted by the index of the radical or by any factor of that index.

280. To reduce a radical to its simplest form.

FIRST.—*When the radical contains a factor that is a perfect power of the same degree as the radical.*

I. Required to simplify $\sqrt{8}$. Also $\sqrt[3]{128}$.

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}.$$

$$\sqrt[3]{128} = \sqrt[3]{64 \times 2} = \sqrt[3]{64} \times \sqrt[3]{2} = 4\sqrt[3]{2}.$$

II. Required to simplify $\sqrt{50a^3b^2 - 75a^2b^3}$.

$$\sqrt{50a^3b^2 - 75a^2b^3} = \sqrt{25a^2b^2(2a - 3b)} = \sqrt{25a^2b^2} \times \sqrt{2a - 3b}$$
$$= 5ab\sqrt{2a - 3b}.$$

Rule.—*Separate the radical into two factors, one of which is the highest perfect power of the same degree as the radical.*

Extract the required root of this factor, and write the result as the co-efficient of the other factor.

EXERCISES.

281. ***Simplify:***

1. $\sqrt{12}$.	8. $\sqrt{45}$.	15. $\sqrt{112m^2n^5}$.
2. $\sqrt{20}$.	9. $\sqrt{63a^3}$.	16. $\sqrt{108a^3x^2}$.
3. $\sqrt{18}$.	10. $\sqrt{54a^2b}$.	17. $\sqrt{125x^6y^3}$.
4. $\sqrt{27}$.	11. $\sqrt{72a^4b^3}$.	18. $\sqrt{98m^3n^4}$.
5. $\sqrt{32}$.	12. $\sqrt{75x^2y^3}$.	19. $\sqrt{147p^5q^7}$.
6. $\sqrt{48}$.	13. $\sqrt{80m^4n^5}$.	20. $\sqrt{192x^7y^5z}$.
7. $\sqrt{50}$.	14. $\sqrt{99a^3b}$.	21. $\sqrt{405x^2y^3z}$.

22.	$\sqrt{500m^5n^3}$.	26.	$\sqrt[3]{192a^6b^3c}$.	30.	$\sqrt[4]{162a^5x^4y}$.
23.	$\sqrt[3]{54a^4}$.	27.	$\sqrt[3]{250m^4n^4}$.	31.	$\sqrt[4]{1250ab^5}$.
24.	$\sqrt[3]{32x^3y^4}$.	28.	$\sqrt[3]{432x^5y^4}$.	32.	$\sqrt[5]{64a^6x^7y}$.
25.	$\sqrt[3]{40m^4n^2}$.	29.	$\sqrt[4]{32a^8b^5c^3}$.	33.	$\sqrt[6]{1458a^6x^{12}}$

34. $\sqrt{3a^2 + 6ab + 3b^2}$. 35. $\sqrt{12a^2 - 36ab + 27b^2}$.

36. $\sqrt{2ax^2 - 4axy + 2ay^2}$.

37. $\sqrt{125a^2b - 200ab + 80b}$.

38. $\sqrt{27ax^2 - 36ax + 12a}$.

39. $\sqrt{12x^2 + 60xy + 75y^2}$.

40. $\sqrt{18a^2 - 84ab + 98b^2}$.

41. $\sqrt{45x^3 - 210x^2y + 245xy^2}$.

42. $\sqrt{5x + 20x^{\frac{1}{2}}y^{\frac{1}{2}} + 20y^{\frac{1}{2}}}$.

43. $\sqrt{8a^4 - 64a^2 + 128}$.

44. $\sqrt[3]{4x^3 - 36x^2 + 108x - 324}$.

45. $\sqrt[3]{3x^3 - 9x^2y + 9xy^2 - 3y^3}$.

SECOND.—*When the radical is a perfect power whose exponent is a factor of the index of the radical.*

III. Required to simplify $\sqrt[6]{49}$. Also $\sqrt[9]{1000}$.

$$\sqrt[6]{49} = \sqrt[6]{7^2} = 7^{\frac{2}{6}} = 7^{\frac{1}{3}} = \sqrt[3]{7}.$$

$$\sqrt[9]{1000} = \sqrt[9]{10^3} = 10^{\frac{3}{9}} = 10^{\frac{1}{3}} = \sqrt[3]{10}.$$

IV. Required to simplify $\sqrt[4]{a^2 - 4a + 4}$.

Also $\sqrt[6]{x^3 - 3x^2y + 3xy^2 - y^3}$.

$$\sqrt[4]{a^2 - 4a + 4} = \sqrt[4]{(a-2)^2} = (a-2)^{\frac{2}{4}}$$
$$= (a-2)^{\frac{1}{2}} = \sqrt{a-2}.$$

$$\sqrt[6]{x^3 - 3x^2y + 3xy^2 - y^3} = \sqrt[6]{(x-y)^3} = (x-y)^{\frac{3}{6}}$$
$$= (x-y)^{\frac{1}{2}} = \sqrt{x-y}.$$

Rule.—*Express the radical with a fractional exponent. Reduce the exponent to its lowest terms. Write the result with a radical sign.*

EXERCISES.

282. ***Simplify the following:***

1.	$\sqrt[4]{9}$.	*9.*	$\sqrt[12]{64}$.	*17.*	$\sqrt[12]{81a^4}$.
2.	$\sqrt[4]{25}$.	*10.*	$\sqrt[14]{128}$.	*18.*	$\sqrt[15]{32a^5b^{10}}$.
3.	$\sqrt[4]{64}$.	*11.*	$\sqrt[9]{343}$.	*19.*	$\sqrt[10]{243x^5y^{15}}$.
4.	$\sqrt[4]{100}$.	*12.*	$\sqrt[6]{216}$.	*20.*	$\sqrt[9]{1000a^3b^6}$.
5.	$\sqrt[6]{100}$.	*13.*	$\sqrt[12]{256}$.	*21.*	$\sqrt[6]{1000x^3y^9}$.
6.	$\sqrt[6]{64}$.	*14.*	$\sqrt[15]{243}$.	*22.*	$\sqrt[12]{1000a^6b^3}$.
7.	$\sqrt[6]{27}$.	*15.*	$\sqrt[6]{125}$.	*23.*	$\sqrt[18]{512a^9b^{18}}$.
8.	$\sqrt[10]{32}$.	*16.*	$\sqrt[9]{216}$.	*24.*	$\sqrt[12]{625x^4y^8}$.

25.	$\sqrt[6]{5^3 \times 3^3a^6b^9}$.	*29.*	$\sqrt[12]{19^2 \times 19^4a^6b^{12}}$.
26.	$\sqrt[9]{3^3 \times 5^3 \times 2^3}$.	*30.*	$\sqrt[10]{10^2 \times 10^3a^5x^{10}}$.
27.	$\sqrt[8]{5^3 \times 5a^4x^8}$.	*31.*	$\sqrt[8]{11^4 \times 7^4a^4b^8c^{12}}$.
28.	$\sqrt[10]{3^5 \times 2^5a^5m^{10}}$.	*32.*	$\sqrt[6]{5^3 \times 7^3 \times 3^3a^6b^3}$.

THIRD.— *When there is a fraction under the radical sign*

V. Required to simplify $\sqrt{\frac{3}{4}}$. Also $\sqrt{\frac{4a}{3bc^3}}$.

$$\sqrt{\tfrac{3}{4}} = \sqrt{\tfrac{1}{4} \times 3} = \sqrt{\tfrac{1}{4}} \times \sqrt{3} = \tfrac{1}{2}\sqrt{3}.$$

$$\sqrt{\frac{4a}{3bc^3}} = \sqrt{\frac{4a}{3bc^3} \times \frac{3b}{3b}} = \sqrt{\frac{12ab}{9b^2c^3}} = \sqrt{\frac{4}{9b^2c^2} \times 3ab}$$

$$= \frac{2}{3bc}\sqrt{3ab}.$$

VI. Required to simplify $2\sqrt[3]{\frac{11}{16}}$. Also $\frac{2a}{3b}\sqrt{\frac{9b^4}{50a^2c}}$.

$$2\sqrt[3]{\tfrac{11}{16}} = 2\sqrt[3]{\tfrac{11}{16} \times \tfrac{8}{8}} = 2\sqrt[3]{\tfrac{88}{128} \times 10} = 2 \times \tfrac{1}{8}\sqrt[3]{10} = \tfrac{1}{4}\sqrt[3]{10}.$$

$$\frac{2a}{3b}\sqrt{\frac{9b^4}{50a^2c}} = \frac{2a}{3b}\sqrt{\frac{9b^4}{50a^2c} \times \frac{2c}{2c}} = \frac{2a}{3b}\sqrt{\frac{9b^4}{100a^2c^2} \times 2c}$$

$$= \frac{2a}{3b} \times \frac{3b^2}{10ac}\sqrt{2c} = \frac{b}{5c}\sqrt{2c}.$$

EXERCISES.

283. *Simplify the following:*

1. $\sqrt{\frac{3}{5}}$.	5. $\sqrt{\frac{11}{16}}$.	9. $\sqrt[3]{\frac{1}{3}}$.
2. $\sqrt{\frac{5}{8}}$.	6. $\sqrt{\frac{11}{50}}$.	10. $\sqrt[3]{\frac{11}{16}}$.
3. $\sqrt{\frac{4}{5}}$.	7. $\sqrt{\frac{3}{7}}$.	11. $\sqrt[3]{\frac{14}{25}}$.
4. $\sqrt{\frac{7}{12}}$.	8. $\sqrt[3]{\frac{1}{2}}$.	12. $\sqrt[3]{\frac{31}{72}}$.

13. $\sqrt[3]{\frac{16}{125}}$.

14. $\sqrt[3]{\frac{8}{27}}$.

15. $\sqrt[4]{\frac{1}{81}}$.

16. $\sqrt[4]{\frac{16}{81}}$.

17. $2\sqrt{\frac{1}{2}}$.

18. $3\sqrt{\frac{2}{3}}$.

19. $5\sqrt{\frac{4}{5}}$.

20. $2\sqrt[3]{\frac{1}{2}}$.

21. $\frac{2}{3}\sqrt[3]{\frac{9}{4}}$.

22. $\frac{5}{6}\sqrt[3]{\frac{4}{5}}$.

23. $\frac{4b}{3a}\sqrt[3]{\frac{5a^4}{16b^2}}$.

24. $\frac{5}{4}\sqrt[3]{\frac{8x^6}{25a^2}}$.

25. $\frac{3}{2}\sqrt{\frac{4a^2b}{5xy^2}}$.

26. $\frac{10}{3}\sqrt{\frac{9mn^2}{10x^2y^3}}$.

27. $5\frac{1}{2}\sqrt{\frac{10ab^2}{11x^2y}}$.

28. $8\frac{2}{3}\sqrt{\frac{9a^2b}{13x^3y^5}}$.

29. $\frac{3a^3b}{x^2y}\sqrt[3]{\frac{5x^3y^4}{6a^4b^3}}$.

30. $\frac{2a}{3b}\sqrt{4\frac{1}{2}b^4}$.

31. $2\frac{1}{4}\sqrt{5\frac{1}{3}a^4}$.

32. $1\frac{2}{5}\sqrt{\frac{25a^3}{28b}}$.

33. $(x-y)\sqrt{\frac{x+y}{x-y}}$.

34. $\frac{a+x}{b}\sqrt{\frac{ab^2}{4(a+x)}}$.

35. $\frac{a^2-x^2}{x}\sqrt{\frac{ax^2}{a-x}}$.

36. $5(x-y)\sqrt{\frac{x^2+2xy+y^2}{5(x^2-y^2)}}$.

284. To introduce the co-efficient of a radical under the radical sign.

I. Required to introduce under the radical signs the co-efficients of $\frac{3}{2}\sqrt[3]{\frac{16}{27}}$, and $\frac{a+1}{a-1}\sqrt{\frac{a-1}{a+1}}$.

$$\tfrac{3}{2}\sqrt[3]{\tfrac{16}{27}} = \sqrt[3]{(\tfrac{3}{2})^3} \times \sqrt[3]{\tfrac{16}{27}} = \sqrt[3]{\tfrac{27}{8} \times \tfrac{16}{27}} = \sqrt[3]{2}.$$

$$\frac{a+1}{a-1}\sqrt{\frac{a-1}{a+1}} = \sqrt{\frac{(a+1)^2}{(a-1)^2}} \times \sqrt{\frac{a-1}{a+1}}$$
$$= \sqrt{\frac{(a+1)^2}{(a-1)^2} \times \frac{a-1}{a+1}}$$
$$= \sqrt{\frac{a+1}{a-1}}.$$

Rule.—*Raise the co-efficient to a power whose exponent is equal to the index of the radical. Multiply the quantity under the radical sign by the power thus obtained.*

EXERCISES.

285. *Introduce under the radical sign the co-efficients of the following:*

1. $5\sqrt{\frac{1}{2}\frac{9}{5}}$.
2. $3\sqrt{5}$.
3. $\frac{2}{3}\sqrt{18}$.
4. $2\frac{2}{3}\sqrt{45}$.
5. $3\frac{3}{4}\sqrt{\frac{3}{4}\frac{2}{5}}$.
6. $3\sqrt[3]{2}$.
7. $5\sqrt[3]{\frac{1}{2}\frac{9}{5}}$.
8. $2\sqrt[4]{2}$.
9. $3\sqrt[3]{2ab}$.
10. $2a^2b\sqrt[3]{ab^2}$.
11. $3ab^2\sqrt{\frac{2}{ab}}$.
12. $2\frac{1}{2}a\sqrt[3]{\frac{64x}{25a^2}}$.
13. $2\frac{2}{3}mn^2\sqrt{5\frac{4}{8}}$.
14. $\frac{2}{3a}\sqrt[4]{7\frac{1}{8}a^5}$.
15. $3x^2y\sqrt[4]{\frac{1}{27x^3y^3}}$.
16. $(x-3)\sqrt{\frac{x+3}{x-3}}$.
17. $\frac{a+b}{a-b}\sqrt{\frac{a-b}{a+b}}$.
18. $(x+3)\sqrt{1-\frac{6}{x+3}}$.
19. $\frac{x+a}{x-a}\sqrt{1-\frac{2a}{x+a}}$.

286. To reduce radicals of different degrees to radicals of the same degree.

I. Required to reduce $\sqrt{3}$, $\sqrt[3]{4}$, and $\sqrt[4]{5}$ to equivalent radicals with the same index.

$$\left.\begin{matrix}\sqrt{3}\\ \sqrt[3]{4}\\ \sqrt[4]{5}\end{matrix}\right\} = \left\{\begin{matrix}3^{\frac{1}{2}}\\ 4^{\frac{1}{3}}\\ 5^{\frac{1}{4}}\end{matrix}\right\} = \left\{\begin{matrix}3^{\frac{6}{12}}\\ 4^{\frac{4}{12}}\\ 5^{\frac{3}{12}}\end{matrix}\right\} = \left\{\begin{matrix}\sqrt[12]{3^6}\\ \sqrt[12]{4^4}\\ \sqrt[12]{5^3}\end{matrix}\right\} = \left\{\begin{matrix}\sqrt[12]{729}\\ \sqrt[12]{256}\\ \sqrt[12]{125}\end{matrix}\right.$$

Rule.—*Write the radicals with fractional exponents. Reduce the exponents to the least common denominator. Express the results by means of the radical sign and raise the quantity under the sign to the indicated power.*

EXERCISES.

287. ***Reduce to equivalent radicals of the same degree.***

1. $\sqrt{3}$ and $\sqrt[3]{4}$.
2. $\sqrt{5}$ and $\sqrt[3]{7}$.
3. $\sqrt[3]{2}$ and $\sqrt[4]{3}$.
4. $\sqrt{5}$ and $\sqrt[4]{10}$.
5. $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[6]{7}$.
6. $\sqrt{2}$, $\sqrt[3]{4}$, and $\sqrt[4]{10}$.
7. $\sqrt[10]{2}$, $\sqrt{3}$, and $\sqrt[5]{4}$.
8. $\sqrt[3]{2}$, $\sqrt[6]{5}$, and $\sqrt[9]{11}$.
9. $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$, and $\sqrt[6]{7}$.
10. $\sqrt{2}$, $\sqrt[5]{5}$, $\sqrt[4]{3}$, and $\sqrt[10]{12}$.
11. $\sqrt[3]{a}$, $\sqrt[4]{b}$, $\sqrt[6]{c}$, and $\sqrt[8]{ax}$.
12. $\sqrt{xy}$, $\sqrt[3]{xy^2}$, $\sqrt[4]{a^2b}$, and $\sqrt[6]{ab^2}$.
13. $\sqrt{a+x}$ and $\sqrt[3]{x-y}$.

14. $\sqrt[4]{m-n}$ and $\sqrt[4]{m+n}$.

15. $\sqrt[3]{x+y}$ and $\sqrt[3]{m+1}$.

16. $\sqrt[3]{x^2-1}$ and $\sqrt[3]{x^2+1}$.

17. $\sqrt{18}$, $\sqrt[3]{16}$, $\sqrt[4]{162}$, and $\sqrt[5]{128}$.

18. $\sqrt{75}$, $\sqrt[3]{54}$, $\sqrt[4]{405}$, and $\sqrt[5]{1458}$.

19. $\sqrt{200}$, $\sqrt[3]{375}$, $\sqrt[4]{5103}$, and $\sqrt[5]{5632}$.

ADDITION AND SUBTRACTION.

288. To add or subtract radicals.

I. Required to find the value of $\sqrt{18}+\sqrt{50}-\sqrt{72}$.

$$\left.\begin{array}{l}+\sqrt{18}\\+\sqrt{50}\\-\sqrt{72}\end{array}\right\}=\left\{\begin{array}{l}+3\sqrt{2}\\+5\sqrt{2}\\-6\sqrt{2}\end{array}\right\}=(+3+5-6)\sqrt{2}=2\sqrt{2}.$$

II. Required to find the value of $6\sqrt{\frac{2}{3}}-2\frac{2}{3}\sqrt{24}+12\sqrt{4\frac{1}{6}}$.

$$\left.\begin{array}{l}+6\sqrt{\frac{2}{3}}\\-2\frac{2}{3}\sqrt{24}\\+12\sqrt{4\frac{1}{6}}\end{array}\right\}=\left\{\begin{array}{l}+2\sqrt{6}\\-5\frac{1}{3}\sqrt{6}\\+10\sqrt{6}\end{array}\right\}=(+2-5\tfrac{1}{3}+10)\sqrt{6}=6\tfrac{2}{3}\sqrt{6}.$$

Rule.—*Reduce each radical to its simplest form. If the radicals are similar, combine their co-efficients by addition and subtraction, as required. Annex the common radical part to the result.*

If the radicals are dissimilar, indicate the required operation by means of signs.

EXERCISES.

289. ***Find the value of the following:***

1. $\sqrt{242} + 3\sqrt{18}$.
2. $\sqrt{147} - \sqrt{75}$.
3. $\sqrt{108} + \sqrt{432}$.
4. $2\sqrt{96} + 5\sqrt{216}$.
5. $5\sqrt[3]{320} - 3\sqrt[3]{135}$.
6. $\sqrt[3]{432} + \sqrt[3]{250}$.
7. $3\sqrt{\frac{2}{5}} - 2\sqrt{\frac{1}{10}}$.
8. $\sqrt[3]{500} + \sqrt[3]{108}$.
9. $\sqrt{605} - \sqrt{405}$.
10. $4\sqrt[3]{\frac{3}{4}} + 10\sqrt[3]{\frac{4\,8}{1\,2\,5}}$.
11. $\sqrt{125} + 5\sqrt{405} - \sqrt{500}$.
12. $6\sqrt{\frac{3}{4}} + 10\sqrt{1\frac{8}{1\,2\,5}} - 14\sqrt{\frac{2\,7}{4\,9}}$.
13. $20\sqrt{\frac{4}{5}} - \sqrt{245} + 35\sqrt{1\frac{4}{5}}$.
14. $8\sqrt{\frac{3}{4}} + \sqrt{60} - 2\frac{1}{5}\sqrt{15} + \sqrt{\frac{3}{5}}$.
15. $\sqrt{243xy^2} + \sqrt{192xy^2} - \sqrt{363xy^2}$.
16. $\sqrt{18a^5} + \sqrt{50a^3} - \sqrt{162a^3}$.
17. $\sqrt{2ax^2 + 4ax + 2a} - \sqrt{2ax^2 - 4ax + 2a}$.
18. $\sqrt[3]{81} - 2\sqrt[3]{24} + \sqrt{28} + 2\sqrt{63}$.
19. $\sqrt[3]{a^7x} + \sqrt[3]{8a^4x^4} + \sqrt[3]{ax^7}$.
20. $\sqrt[3]{x^3y^4 - x^6y^3z} + x\sqrt[3]{y^4 - y^3z}$.

MULTIPLICATION.

290. To multiply radicals of the same degree.

I. Required to multiply $\sqrt{8}$ by $\sqrt{12}$.

$$\sqrt{8} \times \sqrt{12} = \sqrt{8 \times 12} = \sqrt{96} = \sqrt{16 \times 6} = 4\sqrt{6}.$$

II. Required to multiply $\sqrt{3}+5\sqrt{2}$ by $2\sqrt{3}+3\sqrt{2}$. Also, $2\sqrt[3]{3}+3\sqrt[3]{2}$ by $5\sqrt[3]{3}+2\sqrt[3]{2}$.

$$\begin{array}{r} \sqrt{3}+5\sqrt{2} \\ 2\sqrt{3}+3\sqrt{2} \\ \hline 2\times 3+10\sqrt{6} \\ +3\sqrt{6}+15\times 2 \\ \hline 6+13\sqrt{6}+30 \end{array}$$

$$= 36+13\sqrt{6}. \quad \textit{Ans.}$$

$$\begin{array}{r} 2\sqrt[3]{3}+3\sqrt[3]{2} \\ 5\sqrt[3]{3}+2\sqrt[3]{2} \\ \hline 10\sqrt[3]{9}+15\sqrt[3]{6} \\ +4\sqrt[3]{6}+6\sqrt[3]{4} \\ \hline 10\sqrt[3]{9}+19\sqrt[3]{6}+6\sqrt[3]{4}. \end{array} \quad \textit{Ans.}$$

Rule.—*Multiply together the quantities under the radical signs. Write the product as a radical with the same index as its radical factors, and prefix to it the product of the co-efficients.*

Reduce the result to its simplest form.

EXERCISES.

291. ***Find the products of the following:***

1. $\sqrt{5}$ by $\sqrt{50}$.
2. $2\sqrt{3}$ by $3\sqrt{12}$.
3. $4\sqrt{6}$ by $3\sqrt{8}$.
4. $2\sqrt[3]{4}$ by $\sqrt[3]{16}$.
5. $5\sqrt[3]{6}$ by $\sqrt[3]{18}$.
6. $4\sqrt[4]{8}$ by $\sqrt[4]{12}$.
7. $5\sqrt[4]{27}$ by $\sqrt[4]{45}$.
8. $ab\sqrt{ab^2}$ by $\sqrt{abx}$.
9. $2x\sqrt[3]{x^2y}$ by $3y\sqrt[3]{xy^2}$.
10. $5x\sqrt[3]{18a^2b}$ by $3\sqrt[3]{3ab^2x}$.
11. $7ab\sqrt[4]{a^3b^2}$ by $2\sqrt[4]{a^2b^3}$.
12. $2\sqrt[4]{36x^3y^5}$ by $3\sqrt[4]{72x^5y^3}$.

13. $3\sqrt[3]{36ab}$ by $4\sqrt[3]{24a^2b}$.

14. $5\sqrt[3]{75}$ by $3\sqrt[3]{45}$.

15. $\sqrt{2}+\sqrt{3}$ by $\sqrt{2}+\sqrt{3}$.

16. $\sqrt{5}-\sqrt{3}$ by $\sqrt{5}-\sqrt{3}$.

17. $\sqrt{10}+\sqrt{6}$ by $\sqrt{10}-\sqrt{6}$.

18. $\sqrt{5}-2\sqrt{3}$ by $2\sqrt{5}-3\sqrt{3}$.

19. $3\sqrt{7}+2\sqrt{5}$ by $2\sqrt{7}-3\sqrt{5}$.

20. $4\sqrt{3}+2\sqrt{5}$ by $4\sqrt{5}-3\sqrt{3}$.

21. $\sqrt[3]{2}+\sqrt[3]{3}$ by $\sqrt[3]{2}+\sqrt[3]{3}$.

22. $\sqrt[3]{5}+\sqrt[3]{7}$ by $\sqrt[3]{5}-\sqrt[3]{7}$.

23. $2\sqrt[3]{4}+3\sqrt[3]{6}$ by $3\sqrt[3]{4}-2\sqrt[3]{6}$.

24. $a\sqrt{b}+x\sqrt{y}$ by $a\sqrt{b}-x\sqrt{y}$.

25. $a\sqrt[3]{b^2}+x\sqrt[3]{y^2}$ by $a\sqrt[3]{b^2}-x\sqrt[3]{y^2}$.

26. $\sqrt{2}+\sqrt{3}-\sqrt{5}$ by $\sqrt{2}-\sqrt{3}+\sqrt{5}$.

27. $\sqrt{3}-\sqrt{5}+2\sqrt{7}$ by $3\sqrt{3}+2\sqrt{5}-\sqrt{7}$.

28. $\sqrt{x+1}+\sqrt{x-1}$ by $\sqrt{x+1}-\sqrt{x-1}$.

29. $\sqrt{2x+5}+\sqrt{2x-1}$ by $\sqrt{2x+5}-\sqrt{2x-1}$.

30. $\sqrt[3]{3}+2\sqrt[3]{5}-\sqrt[3]{4}$ by $\sqrt[3]{9}-\sqrt[3]{2}$.

31. $\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}$ by $\sqrt[3]{a}-\sqrt[3]{b}+\sqrt[3]{c}$.

32. $\sqrt[3]{4}-\sqrt[3]{16}+\sqrt[3]{12}$ by $\sqrt[3]{4}+\sqrt[3]{16}-\sqrt[3]{2}$.

33. $\sqrt[4]{8}-\sqrt[4]{27}+\sqrt[4]{24}$ by $\sqrt[4]{2}-\sqrt[4]{3}$.

292. To multiply radicals of different degrees.

I. Required to multiply $\sqrt[3]{2}$ by $\sqrt[4]{3}$.

$$\left.\begin{matrix}\sqrt[3]{2}\\ \sqrt[4]{3}\end{matrix}\right\} = \left\{\begin{matrix}2^{\frac{1}{3}}\\ 3^{\frac{1}{4}}\end{matrix}\right\} = \left\{\begin{matrix}2^{\frac{4}{12}}\\ 3^{\frac{3}{12}}\end{matrix}\right\} = \left\{\begin{matrix}\sqrt[12]{16}\\ \sqrt[12]{27}\end{matrix}\right\};$$

$$\sqrt[12]{16} \times \sqrt[12]{27} = \sqrt[12]{16 \times 27} = \sqrt[12]{432}.$$

II. Required to multiply $2\sqrt{3} + 3\sqrt[3]{5}$ by $3\sqrt{3} - 2\sqrt[3]{5}$. Also, $\sqrt{a} + \sqrt[4]{ab} + \sqrt{b}$ by $\sqrt{a} - \sqrt[4]{ab} + \sqrt{b}$.

$$\begin{array}{l} 2\sqrt{3} + 3\sqrt[3]{5} \\ 3\sqrt{3} - 2\sqrt[3]{5} \\ \hline 6 \times 3 + 3\sqrt{3} \times 3\sqrt[3]{5} \\ \qquad - 2\sqrt{3} \times 2\sqrt[3]{5} - 6\sqrt[3]{25} \\ \hline 18 + 5\sqrt[6]{675} \qquad - 6\sqrt[3]{25} \end{array}$$

$$\begin{array}{l} \sqrt{a} + \sqrt[4]{ab} + \sqrt{b} \\ \sqrt{a} - \sqrt[4]{ab} + \sqrt{b} \\ \hline a + \sqrt[4]{a^3b} + \sqrt{ab} \\ \quad - \sqrt[4]{a^3b} - \sqrt{ab} - \sqrt{ab^3} \\ \qquad\qquad + \sqrt{ab} + \sqrt[4]{ab^3} + b \\ \hline a \qquad\qquad + \sqrt{ab} \qquad\qquad + b \end{array}$$

Rule.—*Reduce the radicals to equivalent radicals with the same index, and then perform the operation as in Art.* **290.**

EXERCISES.

293. ***Find the products of the following:***

1. $\sqrt{3}$ by $\sqrt[3]{2}$.
2. $\sqrt[3]{4}$ by $\sqrt[4]{3}$.
3. $\sqrt[3]{2ax}$ by $\sqrt[6]{2ax}$.
4. $3\sqrt{2ax}$ by $2\sqrt[3]{xy}$.
5. $2\sqrt{\frac{1}{2}}$ by $3\sqrt[3]{2}$.
6. $\sqrt[3]{24}$ by $6\sqrt[6]{3}$.
7. $\sqrt{2}$ by $\sqrt[3]{3}$.
8. $\sqrt{5}$ by $\sqrt[3]{4}$.
9. $2\sqrt[3]{6}$ by $4\sqrt{3}$.
10. $5\sqrt{3}$ by $3\sqrt[4]{27}$.
11. $\sqrt[3]{9}$ by $\sqrt[6]{18}$.
12. $3\sqrt[3]{25}$ by $\sqrt{15}$.
13. $a\sqrt{b}$ by $b\sqrt[3]{c}$.
14. $8\sqrt{9}$ by $2\sqrt[3]{3}$.
15. $\sqrt[3]{a-x}$ by $\sqrt{a-x}$.
16. $\sqrt[5]{\frac{3}{7}}$ by $\sqrt[3]{\frac{1}{3}}$.
17. $\sqrt{\frac{2}{5}}$ by $\sqrt[3]{\frac{3}{8}}$.
18. $\sqrt{\frac{3}{5}}$ by $\sqrt[3]{\frac{3}{2}}$.
19. $4\sqrt{ab}$ by $5\sqrt[3]{ab^2}$.
20. $\sqrt[3]{9x^2}$ by $\sqrt{3x}$.
21. $\frac{1}{2}\sqrt{\frac{1}{3}}$ by $\frac{2}{3}\sqrt[3]{\frac{2}{3}}$.
22. $\frac{2}{3}\sqrt[3]{4}$ by $\frac{1}{4}\sqrt{6}$.
23. $\sqrt[5]{a^2x}$ by $3\sqrt{2a^3x}$.
24. $\sqrt[3]{ab^2}$ by $\sqrt{ab}$.
25. $3\sqrt[3]{4a^2}$ by $4\sqrt{2a}$.
26. $5\sqrt{12a}$ by $3\sqrt[3]{16a^2}$.
27. $\frac{1}{3}\sqrt{5}$ by $\frac{1}{3}\sqrt[4]{10}$.
28. $3\sqrt{x} \times \sqrt[6]{y} \times 2\sqrt[4]{x} \times \sqrt[6]{y}$.
29. $3\sqrt[3]{\frac{1}{2}} \times 2\sqrt{\frac{2}{3}} \times 5\sqrt[6]{108}$.
30. $3\sqrt{\frac{2}{3}} \times 4\sqrt[3]{\frac{3}{4}} \times \sqrt[4]{2}$.
31. $\sqrt[3]{a} - \sqrt[6]{b^4}$ by $\sqrt[6]{a^2} + \sqrt[3]{b^2}$.
32. $3\sqrt{2ax}$ by $2\sqrt[6]{3a^3x^2}$.

33. $\sqrt[3]{2} - \sqrt{5}$ by $\sqrt[3]{2} + \sqrt{5}$.

34. $\sqrt[3]{4} + \sqrt{3}$ by $\sqrt[3]{4} + \sqrt{3}$.

35. $2\sqrt[3]{2} + 3\sqrt{3}$ by $5\sqrt[3]{2} - 2\sqrt{3}$.

36. $3\sqrt{7} + \sqrt[3]{5}$ by $\sqrt[3]{7} - \sqrt{5}$.

37. $\sqrt{2} - \sqrt[3]{3} + \sqrt{5}$ by $\sqrt[3]{2} - \sqrt{3}$.

38. $\sqrt[3]{2} + \sqrt[4]{4}$ by $\sqrt[3]{2} - \sqrt[4]{4}$.

39. $\sqrt{2} + \sqrt[5]{3} - \sqrt[10]{4}$ by $\sqrt[5]{2} - \sqrt{3}$.

40. $\sqrt[3]{a} + \sqrt[4]{b} - \sqrt[6]{c}$ by $\sqrt{a} - \sqrt[12]{b}$.

41. $2\sqrt{2} - 5\sqrt[5]{3}$ by $3\sqrt[5]{2} - \sqrt[10]{10}$.

42. $3\sqrt[3]{a^2} - 2\sqrt[4]{a^3}$ by $2\sqrt[12]{x^1} + 3\sqrt[6]{y^3}$.

43. $6\sqrt[6]{8} + 5\sqrt[9]{27}$ by $4\sqrt[12]{64} - 3\sqrt[8]{81}$.

44. $\sqrt[10]{32} + \sqrt[6]{16} + \sqrt[9]{125}$ by $\sqrt[4]{4} - \sqrt[6]{25}$.

45. $\sqrt{2} - \sqrt[6]{9} + \sqrt[4]{25}$ by $\sqrt[3]{3} + \sqrt[10]{32}$.

46. $2\sqrt{a} - 3\sqrt[3]{b} + 5\sqrt{c}$ by $\sqrt[3]{a} - \sqrt{b}$.

47. $\sqrt{2} - \sqrt[3]{3} + \sqrt{5}$ by $\sqrt{2} - \sqrt[3]{3} - \sqrt{5}$.

48. $\sqrt{a} - \sqrt[4]{ab} + \sqrt{b}$ by $\sqrt[4]{a} + \sqrt[4]{b}$.

49. $\sqrt{2} + \sqrt[4]{6} + \sqrt{3}$ by $\sqrt{2} - \sqrt[4]{6} + \sqrt{3}$.

50. $\sqrt[3]{a} + \sqrt[6]{ab} + \sqrt[3]{b}$ by $\sqrt[3]{a} - \sqrt[6]{ab} + \sqrt[3]{b}$.

51. $\sqrt[3]{x} + \sqrt[6]{xy} + \sqrt[3]{y}$ by $\sqrt[6]{x} - \sqrt[6]{y}$.

52. $\sqrt[4]{5} + \sqrt[8]{15} + \sqrt[4]{3}$ by $\sqrt[8]{5} - \sqrt[8]{3}$.

53. $2\sqrt{x} - 5\sqrt[4]{xy} + 3\sqrt{y}$ by $\sqrt{x} + \frac{5}{2}\sqrt[4]{xy} + \frac{1}{4}\frac{9}{2}\sqrt{y}$.

54. $\sqrt[3]{x} - 3\sqrt[6]{xy} - 2\sqrt[3]{y}$ by $\sqrt[3]{x} + 3\sqrt[6]{xy} - 2\sqrt{y}$.

DIVISION.

294. To divide radicals of the same degree.

I. Required to divide $\sqrt{54}$ by $\sqrt{6}$. Also, $6a^2\sqrt{18ax}$ by $2\sqrt{3x}$.

Since to extract any root of the terms of a fraction gives that root of the fraction, it follows that

$$\frac{\sqrt{54}}{\sqrt{6}} = \sqrt{\frac{54}{6}} = \sqrt{9} = 3.$$

Again, $\dfrac{6a^2\sqrt{18ax}}{2\sqrt{3x}} = \dfrac{6a^2}{2}\sqrt{\dfrac{18ax}{3x}} = 3a^2\sqrt{6a}.$

II. Required to divide $\frac{2}{3}\sqrt{\frac{3}{4}}$ by $\frac{3}{4}\sqrt{\frac{2}{3}}$.

$$\frac{\frac{2}{3}\sqrt{\frac{3}{4}}}{\frac{3}{4}\sqrt{\frac{2}{3}}} = \frac{\frac{1}{3}\sqrt{3}}{\frac{1}{4}\sqrt{6}} = \frac{4}{3}\sqrt{\tfrac{1}{2}} = \tfrac{2}{3}\sqrt{2}.$$

The division can also be performed without reducing the quantities to their simplest form. Thus,

$$\frac{\frac{2}{3}\sqrt{\frac{3}{4}}}{\frac{3}{4}\sqrt{\frac{2}{3}}} = \tfrac{8}{9}\sqrt{\tfrac{9}{8}} = \tfrac{8}{9}\sqrt{\tfrac{18}{16}} = \tfrac{8}{9}\sqrt{\tfrac{9}{16}\times 2} = \tfrac{8}{9}\times\tfrac{3}{4}\sqrt{2} = \tfrac{2}{3}\sqrt{2}.$$

Rule.—*Divide the co-efficient of the dividend by the co-efficient of the divisor for the co-efficient of the quotient.*

Divide the radical part of the dividend by the radical part of the divisor for the radical part of the quotient. Reduce the quotient to its simplest form.

EXERCISES.

295. ***Divide:***

1. $\sqrt{160}$ by $2\sqrt{8}$.
2. $24\sqrt{2}$ by $2\sqrt{8}$.
3. $10\sqrt{72}$ by $5\sqrt{6}$.
4. $\sqrt{125a^3x^2y}$ by $\sqrt{5a^2xy}$.
5. $6\sqrt{10}$ by $\sqrt{15}$.
6. $\sqrt[3]{a^4b^2c}$ by $\sqrt[3]{ab}$.
7. $\sqrt[3]{a^2-b^2}$ by $\sqrt[3]{a-b}$.
8. $\sqrt{a^2-x^2}$ by $a-x$.
9. $3\sqrt[4]{20}$ by $\sqrt[4]{5}$.
10. $6\sqrt[3]{81}$ by $\sqrt[3]{48}$.
11. $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$.
12. $\frac{1}{2}\sqrt[4]{25}$ by $\frac{1}{3}\sqrt[4]{10}$.
13. $\frac{1}{3}\sqrt{41\frac{1}{3}} + 8\sqrt{12\frac{1}{3}}$ by $\sqrt{98}$.
14. $12\sqrt[3]{\frac{1}{3}} + 2\sqrt[3]{18}$ by $\sqrt[3]{72}$.
15. $12\sqrt[3]{60}$ by $8\sqrt[3]{3\frac{1}{4}} + \sqrt[3]{240}$.
16. $2\sqrt{32} + 4\sqrt{2}$ by $\sqrt{18}$.
17. $\sqrt{3} + 6\sqrt{\frac{1}{3}}$ by $3\sqrt{2}$.
18. $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\sqrt{2} + 3\sqrt{\frac{1}{2}}$.

296. To divide radicals of different degrees.

I. Required to divide $\sqrt[6]{72}$ by $\sqrt{2}$. Also, $\sqrt{3a^2y^3}$ by $\sqrt[3]{2ay}$.

$$\frac{\sqrt[6]{72}}{\sqrt{2}} = \frac{72^{\frac{1}{6}}}{2^{\frac{1}{2}}} = \frac{72^{\frac{1}{6}}}{(2^3)^{\frac{1}{6}}} = \frac{\sqrt[6]{72}}{\sqrt[6]{8}} = \sqrt[6]{9} = \sqrt[3]{3}.$$

$$\sqrt{3a^2y^3} = \sqrt[6]{27a^6y^9}, \text{ and } \sqrt[3]{2ay} = \sqrt[6]{4a^2y^2}.$$

$$\frac{\sqrt{3a^2y^3}}{\sqrt[3]{2ay}} = \frac{\sqrt[6]{27a^6y^9}}{\sqrt[6]{4a^2y^2}} = \sqrt[6]{\frac{27a^4y^7}{4}} = \frac{y}{2}\sqrt[6]{432a^4y}.$$

Rule.—*Reduce the radicals to equivalent radicals of the same degree. With the results perform the division in accordance with the rule in the preceding case.*

NOTE.—The division of one polynomial expression by another is most easily performed by expressing the radicals by means of fractional exponents, and then proceeding as in the division of polynomials.

II. Required to divide $\sqrt[3]{16} - \sqrt[6]{27}$ by $\sqrt[3]{4} + \sqrt[4]{27}$.

$$\frac{\sqrt{16} - \sqrt[4]{27}}{\sqrt[3]{4} + \sqrt[6]{27}} = \frac{2^{\frac{4}{2}} - 3^{\frac{3}{4}}}{2^{\frac{2}{3}} + 3^{\frac{3}{6}}} = 2^{\frac{4}{2}} - 3^{\frac{3}{6}}.$$

EXERCISES.

297. ***Divide:***

1. $\sqrt[6]{72}$ by $\sqrt{2}$.
2. $\sqrt{3}$ by $\sqrt[3]{6}$.
3. $\sqrt[6]{8a^2b^3}$ by $\sqrt[4]{2ab^2}$.
4. $\sqrt{30}$ by $\sqrt[3]{60}$.
5. $\sqrt{\frac{2}{3}}$ by $\sqrt[3]{\frac{2}{5}}$.
6. $1\frac{1}{3}\sqrt[3]{2\frac{2}{5}}$ by $\frac{2}{3}\sqrt{1\frac{1}{3}}$.
7. $\sqrt[6]{64a^3b^3}$ by $\sqrt[3]{4a^2b}$.
8. $\sqrt{8}$ by $\sqrt[4]{12}$.
9. $\sqrt{\frac{2}{3}}$ by $\sqrt[3]{\frac{1}{3}}$.
10. $\sqrt[6]{12a^2x^4}$ by $\sqrt[4]{2ax^2}$.
11. $\sqrt[3]{2a^2x^3}$ by $\sqrt[5]{2a^3x^3}$.
12. $125\sqrt{x^3y^3}$ by $25\sqrt[3]{x^2y^2}$.
13. $6\sqrt[5]{8}$ by $2\sqrt[3]{2}$.
14. $\sqrt[5]{18}$ by $\sqrt{2}$.
15. $\sqrt[3]{9a^2b^4}$ by $\sqrt[4]{27a^2b^5}$.
16. $5\sqrt[6]{500}$ by $\sqrt{5}$.
17. $6\sqrt[6]{128}$ by $\sqrt[4]{4}$.
18. $3\sqrt[6]{392}$ by $4\sqrt[10]{32}$.
19. $\sqrt[3]{a^2} - \sqrt[5]{b^4}$ by $\sqrt[3]{a} - \sqrt[5]{b^2}$.
20. $\sqrt[3]{x^2} + 2\sqrt[12]{x^4y^3} + \sqrt{y}$ by $\sqrt[3]{x} + \sqrt[4]{y}$.
21. $\sqrt[3]{a^2} + \sqrt[6]{a^2b^3} + b$ by $\sqrt[3]{a} - \sqrt[12]{a^6b^3} + \sqrt{b}$.

SIMPLE EQUATIONS CONTAINING RADICALS.

298. To solve a simple equation containing radicals.

I. Given $\sqrt{3x+4} = 4$; to find the value of x.

Squaring, $3x + 4 = 16.$

Whence, $x = 4.$

II. Given $\sqrt{12+x} = 2 + \sqrt{x}$; to find the value of x.

Squaring, $12 + x = 4 + 4\sqrt{x} + x.$

Transposing and collecting terms,

$$4\sqrt{x} = 8.$$

Dividing by 4, $\sqrt{x} = 2.$

Squaring, $x = 4.$

III. Given $\sqrt{x}+\sqrt{x-7} = \frac{21}{\sqrt{x-7}}$; to find the value of x.

Multiplying by $\sqrt{x-7}$,

$$\sqrt{x^2-7x} + x - 7 = 21.$$

Transposing and uniting, $\sqrt{x^2-7x} = 28 - x.$

Squaring, $x^2 - 7x = 784 - 56x + x^2.$

Transposing and uniting, $49x = 784.$

Dividing by 49, $x = 16.$

IV. Given $\sqrt{2x} + 2 = \sqrt{3x + 4}$; to find the value of x.

Squaring, $2x + 4\sqrt{2x} + 4 = 3x + 4.$

Transposing and uniting, $4\sqrt{2x} = x.$

Squaring, $32x = x^2.$

Dividing by x, $x = 32.$

V. Given $\sqrt{1 + \sqrt{3 + \sqrt{6x}}} = 2$; to find the value of x.

Squaring, $1 + \sqrt{3 + \sqrt{6x}} = 4.$

Transposing, $\sqrt{3 + \sqrt{6x}} = 3.$

Squaring, $3 + \sqrt{6x} = 9.$

Transposing, $\sqrt{6x} = 6.$

Squaring, $6x = 36.$

Whence, $x = 6.$

VI. Given $\dfrac{10x - 16}{\sqrt{10x} + 4} + 2 = \dfrac{\sqrt{10x} + 6}{2}.$

Dividing numerator of first fraction by its denominator,

$$\sqrt{10x} - 4 + 2 = \frac{\sqrt{10x} + 6}{2}.$$

Clearing of fractions and transposing,

$$\sqrt{10x} = 10.$$

Whence, by squaring, etc., $x = 10.$

EXERCISES.

299. ***Find the values of x in the following:***

1. $\sqrt{x+9} = 4.$
2. $\sqrt{4x+1} = 3.$
3. $\sqrt{x-9} = 2.$
4. $\sqrt{x^2-144} = x-4.$
5. $\sqrt{x^2+5x+3} = x+2.$
6. $\sqrt{x+9} = 2\sqrt{x}-3.$
7. $\sqrt{x+16} = \sqrt{x}+2.$
8. $\sqrt{x-16} = 8-\sqrt{x}.$
9. $\sqrt{x-32}+\sqrt{x} = 16.$
10. $\sqrt{x} = \sqrt{2}+\sqrt{x-2}.$
11. $\sqrt{x^2-7} = 7-x.$
12. $\sqrt{4x+9} = 2\sqrt{x}+1.$
13. $\frac{1}{\sqrt{x}}-\sqrt{x} = \sqrt{x+1}.$
14. $\sqrt{5+\sqrt{x-4}} = 3.$
15. $\sqrt{4x^2+2x-16} = 2x-2.$
16. $\sqrt{x-5}-7 = \sqrt{x-12}.$
17. $\sqrt{x-5}+\sqrt{x+7} = 6.$
18. $\sqrt{5+x}+\sqrt{x} = \frac{15}{\sqrt{5+x}}.$
19. $\sqrt{2x-1}+\sqrt{2x+4} = 5.$
20. $(\sqrt{x}-2)^2+4\sqrt{x} = 13.$
21. $\sqrt{x}+\sqrt{x-9} = \frac{36}{\sqrt{x-9}}.$
22. $\sqrt{5x+10}-\sqrt{5x} = 2.$
23. $\sqrt{13+\sqrt{7+\sqrt{3+\sqrt{x}}}} = 4.$
24. $\sqrt{21+\sqrt{7+\sqrt{x-4}}} = 5.$

300. An equation expressing the equality of two fractions, or an equation in which one member is fractional and the other entire, may often be elegantly solved by employing one of the following combinations of the terms of equal fractions.

Let a, b, c, and d be four quantities such that

$$\frac{a}{b} = \frac{c}{d}, \quad \text{just as} \quad \frac{4}{2} = \frac{6}{3}.$$

Then will:

1. $\dfrac{a+b}{a} = \dfrac{c+d}{c}$, just as $\dfrac{4+2}{4} = \dfrac{6+3}{6}$.
2. $\dfrac{a+b}{b} = \dfrac{c+d}{d}$, " $\dfrac{4+2}{2} = \dfrac{6+3}{3}$.
3. $\dfrac{a-b}{a} = \dfrac{c-d}{c}$, " $\dfrac{4-2}{4} = \dfrac{6-3}{6}$.
4. $\dfrac{a-b}{b} = \dfrac{c-d}{d}$, " $\dfrac{4-2}{2} = \dfrac{6-3}{3}$.
5. $\dfrac{b-a}{a} = \dfrac{d-c}{c}$, " $\dfrac{2-4}{4} = \dfrac{3-6}{6}$.
6. $\dfrac{b-a}{b} = \dfrac{d-c}{d}$, " $\dfrac{2-4}{2} = \dfrac{3-6}{3}$.
7. $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$, " $\dfrac{4+2}{4-2} = \dfrac{6+3}{6-3}$.
8. $\dfrac{a+b}{b-a} = \dfrac{c+d}{d-c}$, " $\dfrac{4+2}{2-4} = \dfrac{6+3}{3-6}$.

The demonstration of the foregoing will be found in the chapter on Proportion.

Of these eight forms the pupil should use that one which will give the simplest result.

I. Given $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{\sqrt{x}+38}{\sqrt{x}+6}$.

By (4), $\dfrac{24}{\sqrt{x}+4}=\dfrac{32}{\sqrt{x}+6}$.

Dividing numerators by 8,

$$\frac{3}{\sqrt{x}+4}=\frac{4}{\sqrt{x}+6}.$$

Clearing of fractions,

$$4\sqrt{x}+16=3\sqrt{x}+18.$$

Whence, $\sqrt{x}=2$, and $x=4$.

II. Given $\dfrac{\sqrt{a}+\sqrt{a-x}}{\sqrt{a}-\sqrt{a-x}}=\dfrac{1}{a}$.

By (7), $\dfrac{2\sqrt{a}}{2\sqrt{a-x}}=\dfrac{1+a}{1-a}$.

Simplifying, $\dfrac{\sqrt{a}}{\sqrt{a-x}}=\dfrac{1+a}{1-a}$.

Squaring, $\dfrac{a}{a-x}=\dfrac{1+2a+a^2}{1-2a+a^2}$.

By (5), $\dfrac{x}{a}=\dfrac{4a}{(1+a)^2}$.

Clearing of fractions,

$$(1+a)^2x=4a^2.$$

Whence, $x=\dfrac{4a^2}{(1+a)^2}$.

III. Given $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b.$

Writing the second member as $\frac{b}{1}$, (7) gives,

$$\frac{2\sqrt{a+x}}{2\sqrt{a-x}}=\frac{b+1}{b-1}.$$

Simplifying, $$\frac{\sqrt{a+x}}{\sqrt{a-x}}=\frac{b+1}{b-1}.$$

Squaring, $$\frac{a+x}{a-x}=\frac{b^2+2b+1}{b^2-2b+1}.$$

By (7), $$\frac{2a}{2x}=\frac{2b^2+2}{4b}.$$

Simplifying, $$\frac{a}{x}=\frac{b^2+1}{2b}.$$

Whence, $$x=\frac{2ab}{b^2+1}.$$

EXERCISES.

301. ***Find the values of x in the following:***

1. $\dfrac{\sqrt{x}+3}{\sqrt{x}-1}=\dfrac{\sqrt{x}+13}{\sqrt{x}+1}.$

2. $\dfrac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}}=4.$

3. $\dfrac{3\sqrt{x-4}}{\sqrt{x+2}}=\dfrac{3\sqrt{x+15}}{\sqrt{x+40}}.$

4. $\dfrac{1+x+\sqrt{2x+x^2}}{1+x-\sqrt{2x+x^2}} = \dfrac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}-\sqrt{x}}$.

5. $\dfrac{\sqrt{x-2}}{\sqrt{x+3}} = \dfrac{\sqrt{x+1}}{\sqrt{x+21}}$.

6. $\dfrac{\sqrt{2x-7}+\sqrt{x}}{\sqrt{2x-7}-\sqrt{x}} = \dfrac{19}{3}$.

7. $\dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}} = n$.

8. $\dfrac{\sqrt{12x+1}+\sqrt{12x}}{\sqrt{12x+1}-\sqrt{12x}} = 18$.

9. $\dfrac{\sqrt{4x+1}+2\sqrt{x}}{\sqrt{4x+1}-2\sqrt{x}} = 9$.

10. $\dfrac{\sqrt{x-1}}{\sqrt{x+1}} = \dfrac{2(x-1)}{x+1}$.

11. $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2} = \dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$.

12. $\dfrac{\sqrt{9x}-4}{\sqrt{x}+2} = \dfrac{\sqrt{9x}+15}{\sqrt{x}+40}$.

13. $\dfrac{\sqrt{x}+2a}{\sqrt{x}+b} = \dfrac{\sqrt{x}+4a}{\sqrt{x}+3b}$.

14. $\dfrac{\sqrt{ax}-b}{\sqrt{ax}+b} = \dfrac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$.

15. $\dfrac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}} = \dfrac{a}{b}$.

QUADRATIC EQUATIONS.

302. A **quadratic equation** is an equation containing the square, but no higher power, of the unknown quantity.

Thus, $3x^2 - 12 = 0$, $ax^2 + bx = c$, $5x^2 - 3x = x^2 + 52$, are quadratic equations.

303. A **pure** or **incomplete quadratic equation** is an equation containing no power of the unknown quantity except the second.

Thus, $2x^2 - 15 = 35$, $ax^2 = b$, are pure quadratic equations.

304. An **affected** or **complete quadratic equation** is an equation containing both the second and first powers of the unknown quantity.

Thus, $4x^2 + 5x = 51$, $ax^2 + bx + c = 0$, are complete quadratic equations.

305. The **roots** of an equation are the values of its unknown quantities. The term *root* in this sense is to be distinguished from the same term as used in the operation of evolution.

306. **Real roots** of an equation are values of the unknown quantities that can be found either exactly or approximately.

307. Imaginary or **impossible roots** are values of the unknown quantities containing imaginary quantities. Their values cannot be found either exactly or approximately.

308. An equation is **satisfied** or its roots **verified** when to substitute the roots for the unknown quantities in the equation renders the equation identical.

PURE QUADRATIC EQUATIONS.

309. To solve a pure quadratic equation.

I. Given the equation $7x^2 - 159 = 66 - 2x^2$; to find the value of x.

Solution.

$$7x^2 - 159 = 66 - 2x^2.$$

Transposing and uniting terms,

$$9x^2 = 225.$$

Dividing by 9, the co-efficient of x^2,

$$x^2 = 25.$$

Taking the square root of both members,

$$x = \pm 5.$$

II. Given the equation $5x^2 - 23 = 3x^2 - 17$; to find the value of x.

Solution.

$$5x^2 - 23 = 3x^2 - 17.$$

Transposing and uniting terms,

$$2x^2 = 6.$$

Dividing by 2, the co-efficient of x^2,

$$x^2 = 3.$$

Taking the square root of both members,

$$x = \pm \sqrt{3}.$$

III. Required to solve the equation $\frac{x+1}{x} + \frac{3x}{x+1} = 2.$

Solution.

$$\frac{x+1}{x} + \frac{3x}{x+1} = 2.$$

Clearing of fractions,

$$x^2 + 2x + 1 + 3x^2 = 2x^2 + 2x.$$

Transposing and uniting terms,

$$2x^2 = -1.$$

Dividing by 2, the co-efficient of x^2,

$$x^2 = -\tfrac{1}{2}.$$

Taking the square root of both members,

$$x = \pm \sqrt{-\tfrac{1}{2}} = \pm \tfrac{1}{2}\sqrt{-2}.$$

NOTE.—The roots of this equation are *imaginary* or *impossible*.

Rule.—*Clear the equation of fractions, if necessary. Transpose into the first member the terms containing the unknown quantity, and unite them into one term. Transpose the known terms into the second member, and unite them into one term. Divide both members by the co-efficient of the unknown quantity. Take, or indicate, the square root of both members.*

EXERCISES.

310. ***Find the roots of the following equations:***

1. $3x^2 + 8 = 56.$
2. $x^2 - 8 = 28.$
3. $4x^2 - 64 = 0.$
4. $3x^2 - 15 = 83 + x^2.$
5. $2x^2 - 12 = 13 - 3x^2.$
6. $3x^2 - 57 - 4x^2 = 6 - 8x^2.$
7. $2x^2 + 130 = 5x^2 - 17.$
8. $51x^2 - 96 = 39x^2 + 96.$
9. $5x^2 - 27 = 3x^2 + 215.$
10. $9x^2 + 9 = 3x^2 + 63.$
11. $x^2 + 1 = \frac{x^2}{4} + 4.$
12. $x^2 + \frac{2x^2}{3} = \frac{4x^2}{9} + 99.$
13. $\frac{1}{2}x^2 = 14 - 3x^2.$
14. $x^2 + 5 = \frac{10}{3}x^2 - 16.$
15. $8x + \frac{7}{x} = \frac{65x}{7}.$
16. $\frac{3}{4x^2} - \frac{1}{6x^2} = \frac{7}{3}.$
17. $(x + 2)^2 = 4x + 5.$
18. $\frac{3x^2}{4} - \frac{15x^2 + 8}{6} = 2x^2 - 3.$
19. $\frac{x^2}{5} - \frac{x^2 - 10}{15} = 7 - \frac{50 + x^2}{25}.$
20. $\frac{3x^2 - 27}{x^2 + 3} + \frac{90 + 4x^2}{x^2 + 9} = 7.$
21. $\frac{4x^2 + 5}{10} - \frac{2x^2 - 5}{15} = \frac{7x^2 - 25}{20}.$
22. $\frac{10x^2 + 17}{18} - \frac{12x^2 + 2}{11x^2 - 8} = \frac{5x^2 - 4}{9}.$
23. $\frac{14x^2 + 16}{21} - \frac{2x^2 + 8}{8x^2 - 11} = \frac{2x^2}{3}.$

IV. Given $x + \sqrt{x^2 - \sqrt{2 - 4x}} = 1$;

to find the value of x.

Transposing, $\sqrt{x^2 - \sqrt{2 - 4x}} = 1 - x.$

Squaring, $x^2 - \sqrt{2 - 4x} = 1 - 2x + x^2.$

Whence, $\sqrt{2 - 4x} = 2x - 1.$

Squaring, $2 - 4x = 4x^2 - 4x + 1.$

Whence, $4x^2 = 1,$ or $x^2 = \frac{1}{4}.$

Extracting square root, $x = \pm \frac{1}{2}.$

V. Given $\dfrac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2} - x} = \dfrac{b}{c}$; to find the value of x.

By Art. 300, (7), $\dfrac{2\sqrt{a^2 + x^2}}{2x} = \dfrac{b + c}{b - c}.$

Reducing and squaring, $\dfrac{a^2 + x^2}{x^2} = \dfrac{(b + c)^2}{(b - c)^2}.$

Whence,

$$(b^2 + 2bc + c^2)\,x^2 = a^2(b - c)^2 + (b^2 - 2bc + c^2)\,x^2.$$

Omitting b^2x^2 and c^2x^2 from both sides, and transposing,

$$4bcx^2 = a^2(b - c)^2.$$

And $$x^2 = \frac{a^2(b - c)^2}{4bc}.$$

Whence, $$x = \pm \frac{a(b - c)}{2\sqrt{bc}}.$$

24. $\dfrac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}}=4.$

25. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x.$

26. $\dfrac{\sqrt{1+x}}{1+\sqrt{1+x}}=\dfrac{\sqrt{1-x}}{1-\sqrt{1-x}}.$

27. $\dfrac{1}{\sqrt{1-x}+1}+\dfrac{1}{\sqrt{1+x}-1}=\dfrac{1}{x}.$

28. $\dfrac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\dfrac{x}{2a}.$

29. $\dfrac{\sqrt{3x^2+4}+2}{\sqrt{3x^2+4}-2}=3.$

30. $\dfrac{1}{\sqrt{1-x^2}+1}+\dfrac{1}{\sqrt{1-x^2}-1}=\dfrac{1}{x^2}.$

31. $\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}=\dfrac{1}{2}.$

32. $x\sqrt{6+x^2}=1+x^2.$

33. $\dfrac{1}{\sqrt{a-x}+\sqrt{a}}+\dfrac{1}{\sqrt{a+x}-\sqrt{a}}=\dfrac{\sqrt{a}}{x}.$

34. $\dfrac{ax+1+\sqrt{a^2x^2-1}}{ax+1-\sqrt{a^2x^2-1}}=\dfrac{bx}{2}.$

35. $\dfrac{a}{x}+\dfrac{\sqrt{a^2-x^2}}{x}=\dfrac{x}{b}.$

36. $\dfrac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}=b.$

COMPLETE QUADRATIC EQUATIONS.

311. The terms of every complete quadratic equation are of three kinds:

1st. Terms containing the second power of the unknown quantity.

2d. Terms containing the first power of the unknown quantity.

3d. Known terms, called also **absolute** terms.

These may all be united to form an equation of three terms.

Thus, the equation,

$$-5x^2 + 11x - 14 - 3x^2 = 3x - 4 - 6x^2,$$

reduces to $$-2x^2 + 8x - 10 = 0.$$

If the absolute term is transposed to the right member and both sides of the equation are divided by the co-efficient of x^2, the equation reduces to

$$x^2 - 4x = -5.$$

312. Hence, every quadratic equation may be reduced to the general form

$$x^2 \pm px = \pm q,$$

in which p and q may be integral or fractional, positive or negative.

313. The square of $x \pm \frac{p}{2}$ is equal to $x^2 \pm px + \frac{p^2}{4}$.

Hence, $x^2 \pm px$ is rendered a perfect square by adding to it $\frac{p^2}{4}$, or the square of half the co-efficient of x.

314. The solution of a complete quadratic equation depends essentially upon the fact shown in the preceding article, viz. :

Any binomial of the general form $x^2 \pm px$ may be rendered a perfect square by adding to it the square of half the co-efficient of x.

315. To solve a complete quadratic equation of the form $x^2 \pm px = \pm q$.

I. Required to solve the equation,

$$x^2 - 10x + 9 = 0.$$

Transposing, $$x^2 - 10x = -9.$$

Adding 5^2 to both sides,

$$x^2 - 10x + 25 = 25 - 9 = 16.$$

Extracting the square root,

$$x - 5 = \pm 4.$$

Transposing, $$x = 5 \pm 4 = 9, \text{ or } 1.$$

II. Required to solve the equation,

$$3x^2 + 5x - 100 = 0.$$

Transposing, $$3x^2 + 5x = 100.$$

Dividing by 3, $$x^2 + \tfrac{5}{3}x = \tfrac{100}{3}.$$

Adding $(\frac{5}{6})^2$, $x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{1200}{36} + \frac{25}{36} = \frac{1225}{36}$.

Extracting square root, $x + \frac{5}{6} = \pm \frac{35}{6}$.

Transposing, $x = \pm \frac{35}{6} - \frac{5}{6} = \frac{30}{6}$ or $-\frac{40}{6}$

$= 5$ or $-6\frac{2}{3}$.

III. Required to solve the equation,

$$x^2 - 8x + 19 = 0.$$

Transposing, $x^2 - 8x = -19$.

Adding 4^2, $x^2 - 8x + 16 = 16 - 19 = -3$.

Extracting square root, $x - 4 = \pm \sqrt{-3}$.

Transposing, $x = 4 \pm \sqrt{-3}$.

IV. Required to solve the equation,

$$\frac{2}{x-1} + \frac{5}{x^2-1} = \frac{7}{16}.$$

Clearing of fractions,

$$32x + 32 + 80 = 7x^2 - 7.$$

Transposing, $-7x^2 + 32x = -119$.

Dividing by -7, $x^2 - \frac{32}{7}x = 17$.

Adding $(\frac{16}{7})^2$, $x^2 - \frac{32}{7}x + \frac{256}{49} = 17 + \frac{256}{49} = \frac{1088}{49}$.

Extracting square root, $x - \frac{16}{7} = \pm \frac{33}{7}$.

Transposing, $x = \pm \frac{33}{7} + \frac{16}{7} = \frac{49}{7}$ or $-\frac{17}{7}$

$= 7$, or $-2\frac{3}{7}$.

Rule.—I. *Reduce the equation to the form*

$$x^2 \pm px = \pm q.$$

II. *Add the square of half the co-efficient of the second term to each member of the equation.*

III. *Extract the square root of each member, thus producing a simple equation from which the value of the unknown quantity is found by transposition.*

EXERCISES.

316. ***Find the roots of the following equations:***

1. $x^2 - 8x + 5 = 14.$
2. $x^2 - 12x + 30 = 3.$
3. $x^2 - 4x + 4 = 0.$
4. $x^2 + 6x + 9 = 0.$
5. $x^2 + 16x = 17.$
6. $x^2 - 18x + 81 = 100.$
7. $x^2 + 20x + 100 = 64.$
8. $x^2 - 22x = -121.$
9. $x^2 - 169 = 24x - 144.$
10. $2x^2 - 2x + 1 = x^2 + 9.$
11. $4x^2 - 4x + 4 = 3x^2 + 25.$
12. $6x^2 - 6x + 9 = 5x^2 + 1.$
13. $9x^2 - 8x + 16 = 8x^2 + 25.$
14. $x^2 - 4x + 1 = 9 - 2x.$
15. $4x^2 + 9x + 16 = 4 - 7x.$

16. $5x^2 - 6x + 4 = 4x^2 - 2x + 16.$

17. $4x^2 - 4x + 1 = 4.$

18. $4x^2 - 8x + 4 = 9.$

19. $25x^2 - 20x + 4 = 16.$

20. $5x^2 - 12x + 9 = x^2 + 36.$

21. $100x^2 - 36x + 4 = 19x^2 + 49.$

22. $9x^2 + 12x - 25 = 5x^2 + 4x - 4.$

23. $25x^2 + 20x - 81 = 9x^2 + 4x - 4.$

24. $\frac{x^2}{2} - \frac{x}{3} + 20\frac{1}{2} = 42\frac{2}{3}.$

25. $\frac{1}{2}x^2 - \frac{1}{3}x + 7 = 7\frac{5}{6}.$

26. $6 - x - \frac{x^2}{8} = x + \frac{x^2}{8} - 2\frac{1}{2}.$

27. $x^2 - \frac{3x}{4} - 5 = 16\frac{1}{4}.$

28. $x^2 + 1\frac{2}{3}x = 14.$

29. $x^2 - \frac{2}{3}x + 14\frac{1}{4} = 16.$

30. $\frac{x}{x-1} = \frac{3}{2} + \frac{x-1}{2}.$

31. $\frac{4x}{x+7} - \frac{x-7}{2x+3} - 2 = 0.$

32. $\frac{48}{x+3} + 5 = \frac{165}{x+10}.$

33. $\frac{8x}{x+2} - \frac{20}{3x} - 6 = 0.$

317. When radicals occur in the equation, it must be cleared of them by squaring. The method by combining the terms of equal fractions, already explained, is sometimes applicable in solving radical equations.

I. Required to solve the equation $\sqrt{5x+1}+8=2x$.

Transposing, $\sqrt{5x+1}=2x-8$.

Squaring, $5x+1=4x^2-32x+64$.

Transposing and uniting,

$$-4x^2+37x=63.$$

Dividing by -4, $x^2-\frac{37}{4}x=-\frac{63}{4}$.

Adding $(\frac{37}{8})^2$,

$$x^2-\tfrac{37}{4}x+(\tfrac{37}{8})^2=-\tfrac{63}{4}+\tfrac{1369}{64}=\tfrac{841}{64}.$$

Extracting square root,

$$x-\tfrac{37}{8}=\pm\tfrac{29}{8}.$$

Transposing, $x=\frac{37}{8}\pm\frac{29}{8}=\frac{66}{8}$, or $\frac{8}{8}$

$=7$, or $2\frac{1}{4}$.

It frequently happens that the roots found for a quadratic equation containing radicals will not both satisfy the equation. Thus, the foregoing equation is satisfied by 7, but not by $2\frac{1}{4}$. This arises from the fact that the square of $+\sqrt{5x+1}$ is the same as the square of $-\sqrt{5x+1}$. One of the roots will satisfy the equation as written, the other will satisfy it when the sign of the radical quantity is changed. Thus,

Substituting 7 in $\sqrt{5x+1}+8=2x$;

$$\sqrt{36}+8=2\times 7;\quad 14=14.$$

Substituting $2\frac{1}{4}$ in

$$-\sqrt{5x+1}+8=2x;$$

$$-\sqrt{\tfrac{49}{4}}+8=2\times 2\tfrac{1}{4};\quad 4\tfrac{1}{2}=4\tfrac{1}{2}.$$

II. Required to solve the equation

$$\frac{\sqrt{8x+1}+\sqrt{5x-1}}{\sqrt{8x+1}-\sqrt{5x-1}}=\frac{3\sqrt{x-1}+\sqrt{4x+9}}{3\sqrt{x-1}-\sqrt{4x+9}}.$$

By Art. 300, (7),

$$\frac{2\sqrt{8x+1}}{2\sqrt{5x-1}}=\frac{6\sqrt{x-1}}{2\sqrt{4x+9}}.$$

Reducing and squaring,

$$\frac{8x+1}{5x-1}=\frac{9x-9}{4x+9}.$$

Whence, $45x^2-54x+9=32x^2+76x+9.$

Transposing and uniting,

$$13x^2-130x=0.$$

Dividing by 13, $x^2-10x=0.$

Adding 5^2, $x^2-10x+25=25.$

Extracting square root,

$$x-5=\pm 5.$$

Transposing, $x=10$, or 0.

III. Required to solve the equation

$$\sqrt{3x-3}+\sqrt{5x-19}=\sqrt{2x+8}.$$

Squaring,

$$3x-3+2\sqrt{(3x-3)(5x-19)}+5x-19=2x+8.$$

Transposing and uniting,

$$2\sqrt{(3x-3)(5x-19)}=30-6x.$$

Dividing by 2,

$$\sqrt{(3x-3)(5x-19)}=15-3x.$$

Squaring, $\quad 15x^2-72x+57=225-90x+9x^2.$

Transposing and uniting,

$$6x^2+18x=168.$$

Dividing by 6, $\quad x^2+3x=28.$

Adding $(\frac{3}{2})^2$, $\quad x^2+3x+(\frac{3}{2})^2=28+\frac{9}{4}=\frac{121}{4}.$

Extracting square root, $\quad x+\frac{3}{2}=\pm\frac{11}{2}.$

Transposing, $\quad x=\pm\frac{11}{2}-\frac{3}{2}=\frac{8}{2}$, or $-\frac{14}{2}$

$$=4, \text{ or } -7.$$

The foregoing equation is satisfied by 4, but not by -7. But while -7 will not satisfy the original equation, it will satisfy the derived equation,

$$\sqrt{(3x-3)(5x-19)}=15-3x.$$

EXERCISES.

318. ***Find the roots of the following equations, and verify them.***

1. $x - 5\sqrt{x} = 14.$

2. $x + \sqrt{x+5} = 7.$

3. $\sqrt{4x+17} + \sqrt{x+1} = 4.$

4. $\sqrt{x^2 - 4x - 9} = 6.$

5. $\sqrt{3x^2 + 7x - 2} = 2.$

6. $\sqrt{5x+10} = 8 - x.$

7. $x + 2\sqrt{x^2 + x + 5} = 14.$

8. $\dfrac{x - \sqrt{x+1}}{x + \sqrt{x+1}} = \dfrac{5}{11}$

9. $\dfrac{2(\sqrt{x}+1)}{\sqrt{x}+4} = \dfrac{4 - \sqrt{x}}{\sqrt{x}}$

10. $\sqrt{10x+6} = 9 - x.$

11. $\sqrt{6x+1} + \sqrt{x+4+\sqrt{6x+1}} = 2.$

12. $\sqrt{x+5} = \dfrac{12}{\sqrt{x+12}}.$

13. $\dfrac{5(3x-1)}{5\sqrt{x}+1} + \dfrac{2}{\sqrt{x}} = 3\sqrt{x}.$

14. $\sqrt{2x+1} + 2\sqrt{x} = \dfrac{21}{\sqrt{2x+1}}.$

15. $\dfrac{3\sqrt{x}-10}{5(x-5)}=\dfrac{1}{20}$.

16. $\dfrac{2x+\sqrt{x}}{2x-\sqrt{x}}=\dfrac{52}{15}-\dfrac{3(2x-\sqrt{x})}{2x+\sqrt{x}}$.

17. $(\sqrt{4x+5})(\sqrt{7x+1})=30$.

18. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}$.

19. $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}$.

20. $x+\frac{1}{2}\sqrt{x-3}=8$.

21. $3\sqrt{112-8x}=19+\sqrt{3x+7}$.

22. $3x+2\sqrt{x}=1$.

23. $\sqrt{2x}-7x+52=0$.

24. $\sqrt{x+9}=2\sqrt{x}-3$.

25. $\dfrac{\sqrt{x^2-16}}{\sqrt{x-3}}+\sqrt{x+3}=\dfrac{7}{\sqrt{x-3}}$.

26. $2\sqrt{x}+\dfrac{2}{\sqrt{x}}=5$.

27. $\sqrt{x+8}-\sqrt{x+3}=\sqrt{x}$.

28. $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}$.

29. $\sqrt{2x+9}+\sqrt{3x-15}=\sqrt{7x+8}$.

30. $\sqrt{x+4}-\sqrt{x}=\sqrt{x+1\frac{1}{2}}$.

31. $\sqrt{x} + \sqrt{x - \sqrt{1-x}} = 1.$

32. $\dfrac{\sqrt{1+x}}{1+\sqrt{1+x}} = \dfrac{\sqrt{1-x}}{1-\sqrt{1-x}}.$

33. $2x + 2\sqrt{x+2} = 3\frac{1}{2}.$

34. $\dfrac{3x-1}{\sqrt{3x}+1} = 1 + \dfrac{\sqrt{3x}-1}{2}.$

35. $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}} = \dfrac{x}{a-x}.$

36. $\sqrt{5a+x} + \sqrt{5a-x} = \dfrac{12a}{\sqrt{5a+x}}.$

37. $\dfrac{\sqrt{a+x}}{\sqrt{a}-\sqrt{a+x}} = \dfrac{\sqrt{a-x}}{\sqrt{x}-\sqrt{a-x}}.$

38. $\sqrt{2x+2} = \sqrt{7x} - 3.$

39. $2\sqrt{x} + \sqrt{4x + \sqrt{7x+2}} = 1.$

40. $\dfrac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x+2a}+\sqrt{x-2a}} = \dfrac{x}{2a}.$

319. The foregoing method of solving a complete quadratic equation requires that the equation shall be reduced to the general form $x^2 \pm px = \pm q$, by dividing it by the coefficient of x^2.

If this division is not performed, the equation remains in the form $ax^2 \pm bx = \pm c$, which is the most general form of a quadratic equation.

320. To solve a complete quadratic equation of the form $ax^2 \pm bx = \pm c$.

I. Required to solve the equation, $3x^2 - 7x = 40$.

The first step is to transform the equation so that the left member may be a trinomial that is a perfect square. The first term might be made a perfect square by multiplying the entire equation by the co-efficient 3, but to avoid fractions in the solution, 4 times 3 is employed as a multiplier. The equation then becomes,

$$4 \times 3^2x^2 - 4 \times 3 \times 7x = 480.$$

Now, if A^2 is assumed as representing the quantity that will make the left member a perfect square when added to it, the equation becomes,

$$36x^2 - 84x + A^2 = 480 + A^2.$$

But twice the product of the square roots of the extremes must be equal to the middle term.

That is, $$2 \times 6x \times A = 84x.$$

Whence, $$A = \frac{84x}{12x} = 7, \quad \text{and} \quad A^2 = 49.$$

Substituting 49 for A^2 in the foregoing equation,

$$36x^2 - 84x + 49 = 529.$$

Extracting the square root of both members,

$$6x - 7 = \pm 23.$$

Whence, $$x = 5, \text{ or } -2\tfrac{1}{3}.$$

It will be noticed that 7, the value of A, is the co-efficient of x in the original equation.

II. Required to solve the equation $ax^2 + bx = c$.

This is the general equation given above.

Multiplying by four times the co-efficient of x^2, or $4a$,

$$4a^2x^2 + 4abx = 4ac.$$

Adding the square of the co-efficient of x,

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2.$$

Extracting square root,

$$2ax + b = \pm \sqrt{4ac + b^2}.$$

Transposing, $$2ax = -b \pm \sqrt{4ac + b^2}.$$

Dividing by co-efficient of x, $$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}.$$

Rule.—I. *Reduce the equation to the general form* $ax^2 \pm bx = \pm c$.

II. *Multiply both members of the equation by four times the co-efficient of* x^2.

III. *Add to each member the square of the co-efficient of* x.

IV. *Extract the square root of each member, and the result will be a simple equation from which the roots of* x *may be found by transposition and division.*

EXERCISES.

321. ***Find the roots of the following equations:***

1. $3x^2 - 15x = -18.$
2. $2x^2 - 6x = 20.$
3. $3x^2 - 22x = -7.$
4. $10x^2 - x = 21.$
5. $12x^2 + 13x = 35.$
6. $2x^2 - x = 6.$
7. $15x^2 + 34x = -15.$
8. $14x^2 + x = 175.$

9. $3x^2 + 2x = 85.$

10. $3x^2 + 4x = 340.$

11. $3x^2 - 14x = -15.$

12. $5x^2 + 6x = 63.$

13. $4x^2 - 6x = 108.$

14. $3x^2 - 2x = 65.$

15. $7x^2 - 20x = 32.$

16. $5x^2 + 4x = 273.$

17. $3x^2 + 4x = 95.$

18. $11x^2 - 100x = -201.$

19. $\frac{x^2}{3} + \frac{4x}{5} = 34\frac{1}{5}.$

20. $\frac{x^2}{5} + 20x + 80 = 3x^2.$

21. $\frac{x}{5} + \frac{5}{x} = \frac{26}{5}.$

22. $11\frac{1}{4}x - 3\frac{1}{2}x^2 = -41\frac{1}{4}.$

23. $\frac{x^2}{81} - \frac{2x}{9} = -5.$

24. $\frac{9x^2}{16} + 3\frac{3}{4}x = -4.$

25. $\frac{4x^2}{9} - 4x = 7.$

26. $\frac{4x^2}{49} + \frac{8x}{21} = 6\frac{2}{3}.$

27. $\frac{x^2}{361} - \frac{12x}{19} = -32.$

322. In reducing literal equations to the form of the general equation, it is sometimes necessary to enclose the co-efficient of x^2 or of x in parentheses.

I. Required to solve the equation, $x^2 + ax - x = a.$

Multiplying by four times the co-efficient of x^2, or 4,

$$4x^2 + 4(a - 1)x = 4a.$$

Adding the square of the co-efficient of x,

$$4x^2 + 4(a - 1)x + (a - 1)^2 = (a^2 - 2a + 1) + 4a$$
$$= (a + 1)^2.$$

Extracting square root,

$$2x + (a - 1) = \pm (a + 1).$$

Transposing, $2x = \pm (a + 1) - (a - 1)$

$= 2$, or $-2a$.

Whence, $x = 1$, or $-a$.

II. Required to solve the equation,

$$acx^2 - (ad - bc)\,x = bd.$$

Multiplying by $4ac$,

$$4a^2c^2x^2 - 4ac\,(ad - bc)\,x = 4abcd.$$

Adding $(ad - bc)^2$, the square of the co-efficient of x,

$$4a^2c^2x^2 - 4ac\,(ad - bc)\,x + (ad - bc)^2 = a^2d^2 - 2abcd + b^2c^2 + 4abcd.$$

Or, $\{2acx - (ad - bc)\}^2 = (ad + bc)^2$.

Extracting square root,

$$2acx - (ad - bc) = \pm (ad + bc).$$

Transposing and uniting,

$$2acx = \pm (ad + bc) + (ad - bc).$$

That is, $2acx = 2ad$, or $-2bc$.

Whence, $x = \frac{2ad}{2ac}$, or $-\frac{2bc}{2ac}$.

Reducing, $x = \frac{d}{c}$, or $-\frac{b}{a}$.

EXERCISES.

323. ***Find the roots of the following equations:***

1. $ax^2 - bx = c.$

2. $cx^2 + 2ax = b.$

3. $(x + a)^2 = 5ax - (x - a)^2.$

4. $10x^2 - 9ax = -9a^2.$

5. $(a + b)\,x^2 + (a - b)\,x = \dfrac{ab}{a + b}.$

6. $\dfrac{2x\,(a - x)}{3a - 2x} = \dfrac{a}{4}.$

7. $\dfrac{1}{x + a} + \dfrac{1}{x + 2a} + \dfrac{1}{x + 3a} = \dfrac{3}{x}.$

8. $(a + b)\,x^2 - cx = \dfrac{ac}{a + b}.$

9. $(a - c)\,x^2 - bx = c.$

10. $\dfrac{2c^2}{d^2} + \dfrac{ac}{d} - (a - b)\,(2c + ad)\,\dfrac{x}{d}$
$= (a + b)\,\dfrac{cx}{d} - (a^2 - b^2)\,x^2.$

11. $abx^2 + \dfrac{3a^2x}{c} + \dfrac{b^2x}{c} = \dfrac{(3a + 2b)\,(2a - b)}{c^2}.$

12. $a^2x^2 - 2a^3x + a^4 - 1 = 0.$

13. $(3a^2 + b^2)\,(x^2 - x + 1) = (3b^2 + a^2)\,(x^2 + x + 1).$

324. To write by inspection the roots of any complete quadratic equation in the form $x^2 \pm px = \pm q$.

Since, in this equation, p and q may be either positive or negative, the equation may have any one of four forms. These forms, with their roots, are :

	EQUATIONS.	ROOTS.
1.	$x^2 + px = q.$	$-\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$
2.	$x^2 - px = q.$	$\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$
3.	$x^2 + px = -q.$	$-\frac{p}{2} \pm \sqrt{-q + \frac{p^2}{4}}.$
4.	$x^2 - px = -q.$	$\frac{p}{2} \pm \sqrt{-q + \frac{p^2}{4}}.$

Hence, *The value of x in any complete quadratic in the form $x^2 \pm px = \pm q$ is equal to half the co-efficient of x taken with the opposite sign, $\pm$ the square root of the sum of the square of this half co-efficient and the right member of the equation.*

I. Required to write by inspection the roots of

$$x^2 - 5x = -6.$$

Here p equals -5, and q equals -6; and the equation and its roots correspond with (4) above. Hence,

$$x = \tfrac{5}{2} \pm \sqrt{-6 + \tfrac{25}{4}} = \tfrac{5}{2} \pm \tfrac{1}{2} = 3, \text{ or } 2.$$

II. Required to write by inspection the roots of

$$x^2 - \frac{2a^2bx}{a^2 - b^2} = -\frac{a^2b^2}{a^2 - b^2}.$$

$$x = \frac{a^2b}{a^2 - b^2} \pm \sqrt{-\frac{a^2b^2}{a^2 - b^2} + \left(\frac{a^2b}{a^2 - b^2}\right)^2}$$

$$= \frac{a^2b}{a^2 - b^2} \pm \sqrt{\frac{a^2b^4}{(a^2 - b^2)^2}} = \frac{a^2b}{a^2 - b^2} \pm \frac{ab^2}{a^2 - b^2}$$

$$= \frac{ab\,(a \pm b)}{a^2 - b^2} = \frac{ab}{a - b}, \text{ or } \frac{ab}{a + b}.$$

EXERCISES.

325. ***Write, by inspection, the roots of the following:***

1. $x^2 + 7x = -12$.
2. $x^2 - 5x = -6$.
3. $x^2 + 3x = 18$.
4. $x^2 - 5x = 84$.
5. $x^2 - 10x = 11$.
6. $x^2 - 21x = 100$.
7. $x^2 + 13x = 68$.
8. $x^2 + 96x = 400$.
9. $x^2 - 28x = -195$.
10. $x^2 + 20x = 1125$.
11. $x^2 - ax = 6a^2$.
12. $x^2 - \frac{a^2 - b^2}{ab}x = 1$.
13. $x^2 - \frac{5}{6}x = 1$.
14. $x^2 - \frac{c\,(a^2 - b^2)}{ab}x = c^2$.
15. $x^2 - 1\frac{5}{12}x = -\frac{1}{2}$.
16. $x^2 + 19x = 290$.
17. $x^2 - 2ax = \frac{1 - a^4}{a^2}$.
18. $x^2 - \frac{2cx}{7} = \frac{145c^2}{63}$.
19. $x^2 - 2ax = b^2 - a^2$.
20. $x^2 - \left(\frac{mn}{m + n} - \frac{m + n}{m - n}\right)x = \frac{mn}{m - n}$.

326. To write by inspection the roots of any complete quadratic equation in the form $ax^2 \pm bx = \pm c$.

This is the most general form of the complete quadratic equation. It includes the form $x^2 \pm px = \pm q$, since a may have any value except zero. When a is 1, the equation takes the form of $x^2 \pm px = \pm q$.

EQUATIONS.	ROOTS.
1. $ax^2 + bx = c.$	$\frac{-b \pm \sqrt{4ac + b^2}}{2a}.$
2. $ax^2 - bx = c.$	$\frac{b \pm \sqrt{4ac + b^2}}{2a}.$
3. $ax^2 + bx = -c.$	$\frac{-b \pm \sqrt{-4ac + b^2}}{2a}.$
4. $ax^2 - bx = -c.$	$\frac{b \pm \sqrt{-4ac + b^2}}{2a}.$

Hence, *The value of x in any complete quadratic equation in the form $ax^2 \pm bx = \pm q$ is equal to the co-efficient of x taken with the opposite sign, $\pm$ the square root of the sum of this co-efficient and four times the product of the second member by the co-efficient of x^2, all divided by twice the co-efficient of x^2.*

Required to write by inspection the roots of

$$13x^2 + 56x = 605.$$

Here $a = 13$, $b = 56$, and $c = 605$.

Hence,
$$x = \frac{-56 \pm \sqrt{31460 + 3136}}{26}$$
$$= \frac{-56 \pm 186}{26} = 5, \text{ or } -\frac{121}{13}.$$

EXERCISES.

327. ***Write, by inspection, the roots of the following:***

1. $6x^2 - 19x = -15.$
2. $3x^2 + x = 14.$
3. $15x^2 - 2x = 45.$
4. $35x^2 + 152x = -165.$
5. $3x^2 - x = 44.$
6. $6x^2 + 5x = 1000.$
7. $18x^2 + 13x = 21.$
8. $35x^2 + x = 88.$
9. $6x^2 + 7\frac{1}{2}x = -2\frac{1}{2}.$
10. $10\frac{1}{14}x^2 + 21\frac{1}{14}x = 40\frac{1}{14}.$
11. $6x^2 - 10ax + 9bx = 15ab.$
12. $mnx^2 + (n^2 - m^2)\,x = mn.$
13. $\frac{x^2}{2} + \frac{4x}{15} = \frac{2}{5}.$
14. $\frac{6}{x} - \frac{5\,(x-1)}{4} + \frac{x}{6} = 0.$
15. $\frac{5x\,(x+1)}{21} - \frac{2x^2 + x - 1}{7} = \frac{4\,(x+1)}{35}.$
16. $10x^2 - 19\frac{3}{4}x = -6.$
17. $26x^2 - 6\frac{1}{2}x = 91.$
18. $8x^2 + 36\frac{7}{10}x = 10\frac{1}{5}.$
19. $abx^2 + adx - bcx = cd.$
20. $2acx^2 - bcx = 2adx - bd.$
21. $m^2n^2x^2 + 2mn^2x = m^2 - n^2.$
22. $acx^4 - bc^2x + ax = bc.$

PROBLEMS LEADING TO PURE QUADRATICS.

328. I. Find two numbers that are to each other as 3 to 4, and the difference of their squares is 252.

Let $3x =$ the smaller number,

and $4x =$ the larger number.

Then $16x^2 - 9x^2 = 252$;

or $7x^2 = 252$;

and $x^2 = 36$;

whence, $x = \pm 6$.

Hence, $3x = \pm 18$, and $4x = \pm 24$.

1. Find two numbers that are to each other as 2 to 3, and the difference of their squares is 125.

2. Mr. Brown's age is to his wife's age as 6 to 5, and the sum of the squares of their ages is 1525. What are their ages?

3. The floor of a hall is $2\frac{1}{2}$ times as long as it is wide, and the area of the floor is 4000 square feet. What are its dimensions?

4. A merchant bought a piece of cloth for \$163.35. The number of yards in the piece was to the number of dollars per yard as 15 to 4. Find the price per yard and the number of yards.

5. The product of two numbers is 126 and the quotient of the greater divided by the less is $3\frac{1}{2}$. What are the numbers?

6. Find three numbers that are to each other as 2, 3, and 5, and the sum of their squares is 1862.

7. During a certain year I earned $1400. The number of weeks I worked was to the number of dollars I earned per week as 8 to 7. How many weeks did I work and what was my pay per week ?

8. Three times the cube of a certain number diminished by 5 times the number is equal to 7 times the number. Find the number.

9. Find a number such that one third of the product of its seventh and eighth parts shall be $298\frac{2}{3}$.

10. Joseph bought 80 apples and pears. The apples cost twice as much as the pears; but if he had bought as many apples as he did pears, and as many pears as he did apples, his apples would have cost 10 cents, and his pears 45 cents. How many did he buy of each ?

11. A body of soldiers was marching with 5 men more in depth than in front. By another arrangement, the front line contained 845 men more than before. By this disposition there were only 5 lines. What was the number of men ?

12. A man loaned some money at 6 per cent., and found that if he multiplied the principal by the number representing the interest for 8 months, the product would be $900. What was the principal ?

13. The rods in the length and breadth of a rectangular field are in the ratio of 4 to 3, and its diagonal is 100 rods. How many acres in the field ?

14. Three equal square lots contain 193 square rods less than a square lot 25 rods on each side. What is the length of each side of the three square lots ?

15. There are two numbers whose sum is 17, and the less divided by the greater is to the greater divided by the less as 64 to 81. What are the numbers ?

PROBLEMS LEADING TO COMPLETE QUADRATICS.

329. *1.* Divide the number 56 into two such parts that their product shall be 640.

2. The difference between two numbers is 6, and the sum of their squares is 50. What are the numbers?

3. A man planted 8400 trees, at equal distances, in the form of a rectangle. There were 50 trees more in front than in depth. What was the number in front?

I. A man divided \$144 equally among a certain number of persons. If the number of persons had been 2 less, each person would have received one dollar more. How many persons were there?

Let x = the number of persons,

then $\frac{144}{x}$ = the number of dollars each received,

and $\frac{144}{x-2}$ = the number of dollars each received when there were 2 persons less.

Hence, $$\frac{144}{x-2} - 1 = \frac{144}{x}.$$

Clearing of fractions,

$$144x - x^2 + 2x = 144x - 288.$$

Transposing and uniting,

$$x^2 - 2x = 288.$$

Whence, $$x = 18 \text{ or } -16.$$

In solving problems involving quadratic equations, negative results are often found. They will not satisfy the conditions of the problem. The negative root, however, will generally become the positive root of a related problem, and the positive will become the negative root. The relation of the following problem to the preceding will illustrate this :

4. A man divided $144 equally among a certain number of persons. If the number of persons had been 2 *more,* each person would have received one dollar *less.* How many persons were there ?

5. 110 bushels of coal were divided among a certain number of poor persons. If each person had received one bushel more, he would have received as many bushels as there were persons. How many persons were there ?

6. A certain number of men were to husk 175 bushels of corn. If 2 of them went away before the work began, those that remained would have each 10 bushels more to husk. How many men were there ?

7. A lady bought a piece of meat for $2.16. If the meat had been one cent a pound dearer, she would have received 3 pounds less for the same money. How many pounds did she buy ?

8. A merchant bought a piece of cloth for $45 and sold it at a profit of 15 cents per yard. Although he gave away 5 yards, he gained $4.50 on the piece. How many yards did he buy and at what price per yard ?

9. A rectangular field whose area is 768 square rods is 8 rods more in length than in width. What are the dimensions of the field ?

10. A man sold a horse for $312.50. The number representing his gain per cent. was $\frac{1}{10}$ as much as the number representing its cost in dollars. How much did it cost ?

11. A is 4 years older than B, and the sum of the squares of their ages is 976. What are their ages?

12. A man bought a number of sheep for $240, and sold them at $6.75 each. He gained in all as much as 5 sheep cost him. How many sheep did he buy?

13. A person sold goods for $24, thereby losing as much per cent. as the goods cost in dollars. What did they cost?

14. A certain number consists of two figures whose sum is 12. The product of the figures is 16 less than the number expressed by the figures in inverse order. What is the number?

15. A merchant bought cloth for $120, and after cutting off 4 yards, he sold the remainder for what the whole cost him. His gain on what he sold was $1 per yard. How many yards did he buy?

16. A grocer has two kinds of sugar. He charges 2 cents a pound more for one than for the other, and sells $2\frac{1}{2}$ pounds more of the cheaper kind for $1 than of the dearer kind. How much is each a pound?

17. A and B can do a piece of work in 20 days. It takes A alone 9 days more to do it than it does B alone. How long does it take each alone to do the work?

18. What are eggs worth a dozen when giving 5 more for 25 cents lowers the price 5 cents a dozen?

19. The side of a square is 110 rods. What are the dimensions of a rectangle whose perimeter is 4 rods more than that of the square and its area 4 square rods less than the area of the square?

20. A certain sum is loaned at 5% per annum. The product of the number of dollars in the principal by the number of dollars in the interest for 3 months is 120 more than a number equal to $2\frac{1}{2}$ times the principal. What is the principal?

21. One of two pipes will fill a cistern in 6 hours less than the other, and both of them will fill it in 4 hours. How long will it take each to fill the cistern ?

22. There are two numbers whose difference is 11, and $\frac{1}{8}$ of their product plus 345 is equal to the square of the greater number. What are the numbers ?

23. A and B set out at the same time to travel 150 miles. A's rate per hour was 2 miles more than B's, and it took him 20 hours less to perform the journey than B required. What were their rates per hour ?

24. A and B together can do a certain work in 10 days. After working together for 4 days, A is taken ill, and B finishes the work in 3 days more than A alone would have required to do the whole work. In what time could each alone do the work ?

25. A rectangular field contains one acre and its four sides together measure 308 yards. Find its length and breadth.

26. I can row 6 miles an hour in still water. To row 5 miles up a certain stream takes me 2 hours longer than to row the same distance down the stream. What is the rate of the current ?

27. A merchant sells two casks of wine for $610. One cask contains 10 gallons more than the other, and the price of each wine is in dimes the same as the number of gallons in the cask that contains it. How many gallons are there in each cask ?

28. There are two vessels, one of which sails two miles an hour faster than the other. They start at the same time, the faster vessel to make a voyage of 1152 miles, the slower a voyage of 720 miles. The slower vessel reaches its destination one day sooner than the faster. Find the rate per hour of the faster vessel.

29. A man rows 48 miles up a stream and back in 20 hours. What is his rate of rowing in still water if the rate of the current is 1 mile per hour?

30. The length of a rectangular field is 4 rods more than its width, and its area is 9 acres. Find its dimensions.

31. A grocer bought some chickens for \$18, and some turkeys for \$48. There were 19 more turkeys than chickens, and they cost 35 cents more each. How many of each did he buy?

EQUATIONS IN THE QUADRATIC FORM.

330. An **equation in the quadratic form** is an equation that contains an absolute term and two powers of the unknown quantity, of which one exponent is twice as great as the other.

Thus, $x^4 - 4x^2 = 45$, $x^6 - 5x^3 = 24$, and $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = 88$, are equations in the quadratic form.

331. In an equation in the quadratic form a polynomial may be used as if it were a single letter.

Thus, $(x^2 - 3)^2 + 5(x^2 - 3) = 66$, and $(x^2 + 5x - 7)^2 + 3(x^2 + 5x - 7) = 70$, are equations in the quadratic form.

332. To solve an equation in the quadratic form.

I. Required to solve the equation $x^4 - 16x^2 = 225$.

Completing the square,

$$x^4 - 16x^2 + 64 = 289.$$

Extracting square root, $x^3 - 8 = \pm 17$.

Transposing, $x^3 = 25$, or -9.

Extracting square root, $x = \pm 5$, or $\pm 3\sqrt{-1}$.

II. Required to solve the equation $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = 88$.

Completing the square,

$$x^{\frac{2}{3}} + 3x^{\frac{1}{3}} + \tfrac{9}{4} = 88 + \tfrac{9}{4} = \tfrac{361}{4}.$$

Extracting square root,

$$x^{\frac{1}{3}} + \tfrac{3}{2} = \pm \tfrac{19}{2}.$$

Transposing, $x^{\frac{1}{3}} = 8$, or -11.

Extracting cube root, $x^{\frac{1}{3}} = 2$, or $\sqrt[3]{-11}$.

Raising to the 4th power,

$$x = 16, \text{ or } \sqrt[3]{-11^4}.$$

III. Required to solve the equation $x^{-3} - 6x^{-\frac{3}{2}} = 16$.

Completing the square,

$$x^{-3} - 6x^{-\frac{3}{2}} + 9 = 25$$

Extracting square root,

$$x^{-\frac{3}{2}} - 3 = \pm 5.$$

Transposing, $x^{-\frac{3}{2}} = 8$, or -2.

Extracting cube root, $x^{-\frac{1}{2}} = 2$, or $\sqrt[3]{-2}$.

Squaring, $x^{-1} = 4$, or $\sqrt[3]{4}$.

Taking reciprocals, $x = \frac{1}{4}$, or $\frac{1}{\sqrt[3]{4}} = \frac{1}{2}\sqrt[3]{2}$.

EXERCISES.

333. ***Find the roots of the following equations:***

1. $3x^3 + 42x^{\frac{3}{2}} = 3321.$
2. $x^{\frac{1}{2}} + 7x^{\frac{1}{4}} = 44.$
3. $4x^{\frac{1}{2}} + x^{\frac{1}{4}} = 39.$
4. $3x^6 + 42x^3 = 3321.$
5. $2x^3 - 17x^{\frac{3}{2}} = -8.$
6. $3x^{\frac{3}{4}} - 4x^{\frac{3}{4}} = 7.$
7. $x^{10} + 31x^5 = 32.$
8. $3x + 2x^{\frac{1}{2}} = 1.$
9. $x^4 + 4x^2 = 12.$
10. $5x^{\frac{1}{3}} + x^{\frac{1}{6}} = 22.$
11. $x^3 - x^{\frac{3}{2}} = 56.$
12. $x^{\frac{1}{2}} + x^{\frac{3}{4}} = 756.$
13. $2x^{\frac{3}{2}} - 6x^{\frac{1}{2}} = 20.$
14. $x^{\frac{1}{2}} + \sqrt{2} = \dfrac{2}{x^{\frac{1}{2}} - \sqrt{2}}.$
15. $3x^{\frac{1}{2}} + 7x^{\frac{3}{4}} = 76$

IV. Required to solve $(3x^2 - 4x)^2 + 6(3x^2 - 4x) = 40.$

Completing the square,

$$(3x^2 - 4x)^2 + 6(3x^2 - 4x) + 9 = 49.$$

Extracting square root,

$$3x^2 - 4x + 3 = \pm 7.$$

Transposing, $3x^2 - 4x = 4,$ or $-10.$

Multiplying by 4 times the coefficient of x^2,

$$36x^2 - 48x = 48, \text{ or } -120.$$

Completing the square,

$$36x^2 - 48x + 16 = 64, \text{ or } -104.$$

Extracting square root, $6x - 4 = \pm 8$, or $\pm \sqrt{-104}$.

Transposing, $6x = 12, \ -4, \text{ or } 4 \pm \sqrt{-104}$.

Whence, $x = 2, \ -\frac{2}{3}, \text{ or } \frac{2}{3} \pm \frac{1}{6}\sqrt{-104}$.

Equations like the foregoing are elegantly solved by the use of an auxiliary letter. To illustrate,

Let $3x^2 - 4x = y$.

Then the equation becomes,

$$y^2 + 6y = 40.$$

Whence, $y = 4, \text{ or } -10$.

Hence, $3x^2 - 4x = 4, \text{ or } -10$.

Whence, $x = 2, \ -\frac{5}{3}, \text{ or } \frac{4 \pm \sqrt{-104}}{6}$.

16. $x^2 - 2x + 5 + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 6$.

17. $x + 16 - 3\sqrt{x + 16} = 10$.

18. $(x + 12)^{\frac{1}{2}} + (x + 12)^{\frac{1}{4}} = 6$.

19. $2x^2 - 7x + 6 + 2\sqrt{2x^2 - 7x + 6} = 0$.

20. $(x^2 - 4x + 4)^2 + 5(x^2 - 4x + 4) = 126$.

21. $\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) = \frac{160}{9}$.

22. $x + 21 + (x + 21)^{\frac{1}{2}} = 12$.

23. $(2x + 6)^{\frac{1}{2}} + (2x + 6)^{\frac{1}{4}} = 6$.

24. $x + 6 + \sqrt{x + 6} = 8 + 3\sqrt{x + 6}$.

25. $x + 5 = \sqrt{x + 5} + 6$.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

334. Homogeneous terms are terms of the same degree, or terms that contain the same number of literal factors.

Thus, x^2y, $3xy^2$, $5y^3$, $6xyz$, are homogeneous terms.

335. A **homogeneous equation** is an equation whose terms that contain literal factors are all homogeneous.

Thus, $x^2 + 3xy + y^2 = 101$ is a homogeneous equation.

336. Two simultaneous equations are homogeneous when the terms containing unknown factors are homogeneous in both equations.

Thus, $x^2 + xy + y^2 = 11$, and $3x^2 - 2y^2 = 58$, are two homogeneous simultaneous equations.

337. An equation is symmetrical with respect to two quantities when the equation is not changed by interchanging the quantities.

Thus, $x^2 + 5xy + y^2 = 141$ is symmetrical, since $y^2 + 5yx + x^2 = 141$ is equivalent to the first equation.

But $x^2 - y^2 = 20$ and $3x^2 - 5xy = 100$ are not symmetrical, since they are not equivalent respectively to $y^2 - x^2 = 20$ and $3y^2 - 5yx = 100$.

338. The solution of two quadratic equations involving two unknown quantities generally requires the solution of an equation of the fourth degree. There are certain cases, however, where two simultaneous equations in one or both of

which occur terms of a degree higher than the first, may be solved by means of a final equation in the quadratic form. Most of the examples of this kind belong under one of three cases:

1. *When one equation is simple and the other quadratic.*
2. *When both equations are homogeneous quadratics.*
3. *When both equations are symmetrical.*

339. To solve two simultaneous equations, one of which is simple and the other quadratic.

I. Required to solve the equations,

$$\left\{\begin{array}{rl} 3x - 2y = & 10, \\ 5x^2 - 3xy = & 152. \end{array}\right\} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

From (1), $$x = \frac{10 + 2y}{3}.$$

Substituting in (2),

$$\frac{5\,(10 + 2y)^2}{9} - \frac{3y\,(10 + 2y)}{3} = 152.$$

Clearing of fractions,

$$5\,(100 + 40y + 4y^2) - 9y\,(10 + 2y) = 1368.$$

Or, $$500 + 200y + 20y^2 - 90y - 18y^2 = 1368.$$

Transposing and uniting, $$2y^2 + 110y = 868.$$

Whence, $$y = 7, \text{ or } -62.$$

Substituting in (1), $$\left\{\begin{array}{rl} 3x - 14 = & 10, \\ 3x - (-124) = & 10. \end{array}\right\}$$

Whence, $$x = 8, \text{ or } -38.$$

EXERCISES.

340. ***Find the roots of the following equations:***

1. $5x - y = 13$; $3xy - 2y^2 = 10$.
2. $5x + 2y = 7$; $7x^2 - 8xy = 159$.
3. $x + 4y = 14$; $4x - 2y + y^2 = 11$.
4. $x - y = 12$; $x^2 + y^2 = 74$.
5. $2x + y = 22$; $xy + 2y^2 = 120$.
6. $10x + y = 3xy$; $y - x = 2$.
7. $2x - 3y = 1$; $2x^2 + xy - 5y^2 = 20$.
8. $3x + y = 18$; $x^2 + 2y^2 = 43$.
9. $x + y = 8$; $xy = 2x + y + 2$.
10. $4(x + y) = 5$; $8xy = 3$.
11. $4x = 9y$; $x + y^2 = 25$.
12. $2x - 3y = 1$; $8x^2 - 22xy + 15y^2 = 5$.
13. $x + 3y = 7$; $x^2 + 3xy + y^2 = 11$.
14. $2x - 3y = 2$; $4x^2 + 9y^2 = 100$.
15. $x - y = 3$; $2x^2 + 3y^2 - xy = 31$.
16. $2x + 3y = 16$; $4x^2 - xy = 90$.
17. $2x + 11y = 15$; $4x^2 + 121y^2 = 137$.
18. $x + y = 20$; $x^2 - 2y^2 = 71$.
19. $x + y = 7$; $x^2 + 2y^2 = 34$.
20. $x + y = 2$; $x^3 - 2xy - y^2 = 1$.

341. To solve two simultaneous equations when they are homogeneous quadratics.

I. Required to solve the equations

$$\left.\begin{aligned} 3x^2 - 2xy &= 8, \\ 2y^2 + 3xy &= 110. \end{aligned}\right\} \quad (1) \quad (2)$$

Let $x = vy$.

Then $\left.\begin{aligned} 3x^2 - 2xy &= 8 \\ 2y^2 + 3xy &= 110 \end{aligned}\right\} = \left.\begin{aligned} 3v^2y^2 - 2vy^2 &= 8, \\ 2y^2 + 3vy^2 &= 110. \end{aligned}\right\}$ (3) (4)

From (3), $$y^2 = \frac{8}{3v^2 - 2v}; \quad (5)$$

and from (4), $$y^2 = \frac{110}{2 + 3v}. \quad (6)$$

Hence, $$\frac{8}{3v^2 - 2v} = \frac{110}{2 + 3v}.$$

And $$8 + 12v = 165v^2 - 110v.$$

Transposing, $165v^2 - 122v = 8$.

Whence, $v = \frac{4}{5}$, or $-\frac{2}{33}$.

Substituting $v = \frac{4}{5}$ in (6),

$$y = \pm 5;$$

From $x = vy$, $$x = \pm 4.$$

Substituting $v = -\frac{2}{33}$ in (6),

$$y = \pm \tfrac{11}{3}\sqrt{2}.$$

From $x = vy$, $$x = \mp \tfrac{1}{3}\sqrt{2}.$$

In obtaining the last values of x, it will be observed that taking $v = -\frac{2}{33}$ and $y = +\frac{11}{3}\sqrt{2}$ gives $x = -\frac{1}{3}\sqrt{2}$; and with the same value of v, taking $y = -\frac{11}{3}\sqrt{2}$, gives $x = +\frac{1}{3}\sqrt{2}$. That is, + and −, or ± values of y give

respectively $-$ and $+$ values of x. The sign $\mp$ denotes the precedence of the negative value. In this case the equation can be satisfied only by making $x = -\frac{1}{3}\sqrt{2}$, when $y = +\frac{11}{3}\sqrt{2}$; but when $y = -\frac{11}{3}\sqrt{2}$, x must be $+\frac{1}{3}\sqrt{2}$.

EXERCISES.

342. ***Find the roots of the following equations:***

1. $5x^2 + 3xy = 26$; $3y^2 + 2xy = 7$.
2. $x^2 + xy = 12$; $xy - 2y^2 = 1$.
3. $5x^2 + 6y^2 - 12xy = 2$; $4x^2 + 5y^2 - 10xy = 1$.
4. $4y^2 = 3x^2 + xy$; $x^2 + y^2 = 25$.
5. $x^2 - 2xy - y^2 = 31$; $x^2 + 4xy - 2y^2 = 202$.
6. $10(x^2 + y^2) = 29xy$; $xy = 10$.
7. $4x^2 - 2xy + 3y^2 = 45$; $xy + y^2 = -2$.
8. $x^2 + xy = 15$; $xy - y^2 = 2$.
9. $x^2 + 4y^2 + 4xy = 256$; $3y^2 - x^2 = 39$.
10. $2x^2 + 3xy + y^2 = 20$; $5x^2 + 4y^2 = 41$.
11. $5x^2 - 3xy + 2y^2 = 35$; $2x^2 + 4xy - y^2 = 38$.
12. $x^2 - xy = 6$; $x^2 + y^2 = 61$.
13. $4x^2 - 2xy = 12$; $2y^2 + 3xy = 8$.
14. $x^2 + 2xy + 3y^2 = 17$; $2x^2 + 3xy + 5y^2 = 28$.
15. $6(x^2 - xy) = 7xy - 6y^2$; $x^2 - y^2 = 20$.
16. $x^2 + xy = 77$; $xy - y^2 = 12$.
17. $5x^2 - 3xy = 56$; $5y^2 + xy = 28$.
18. $2x^2 - 3xy = 56$; $xy - y^2 = 15$.
19. $3xy - 4x^2 = 2$; $x^2 + y^2 = 5$.
20. $3x^2 - 3xy + y^2 = 21$; $x^2 - 2xy + 3y^2 = 19$.

343. To solve two simultaneous equations when they are symmetrical quadratics.

In solving examples of this kind, the object should generally be to obtain two simple equations, one of which expresses the sum of the unknown quantities and the other the difference between them. When the equations are symmetrical with respect to the letters, but not with respect to the signs, the solution may often be effected.

Thus, suppose it can be ascertained that

$$x + y = 9,$$

and that $$x - y = 5.$$

Adding gives $$2x = 14,$$

whence, $$x = 7.$$

Subtracting gives $$2y = 4, \quad \text{whence,} \quad y = 2.$$

I. Required to solve the equations,

$$\left\{\begin{array}{rl} x^2 + y^2 = 90, & (1) \\ xy = 27. & (2) \end{array}\right.$$

Adding to (1) twice (2),

$$x^2 + 2xy + y^2 = 144. \qquad (3)$$

Subtracting from (1) twice (2),

$$x^2 - 2xy + y^2 = 36. \qquad (4)$$

Extracting square roots of (3) and (4),

$$x + y = \pm 12, \qquad (5)$$

and $$x - y = \pm 6. \qquad (6)$$

Adding (5) and (6), $$2x = \pm 18.$$

Subtracting (6) from (5), $$2y = \pm 6.$$

Whence, $$x = \pm 9, \quad \text{and} \quad y = \pm 3$$

II. Required to solve the equations,

$$\left.\begin{array}{l} x^2 + y^2 = 65, \\ x + y = 11. \end{array}\right\} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

Squaring (2), $x^2 + 2xy + y^2 = 121.$ (3)

Subtracting (1) from (3), $2xy = 56.$ (4)

Subtracting (4) from (1),

$$x^2 - 2xy + y^2 = 9. \quad (5)$$

Taking square root $x - y = \pm 3.$

But (2) is $x + y = 11.$

Whence, $2x = 14$, or 8; $2y = 8$, or $14.$

And $x = 7$, or 4; $y = 4$, or $7.$

III. Required to solve the equations,

$$\left.\begin{array}{l} x + y = 5, \\ x^3 + y^3 = 65. \end{array}\right\} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

Dividing (2) by (1),

$$x^2 - xy + y^2 = 13. \quad (3)$$

Squaring (1), $x^2 + 2xy + y^2 = 25.$ (4)

Subtracting (3) from (4), $3xy = 12.$ (5)

Dividing (5) by 3, $xy = 4.$ (6)

Subtracting (6) from (3),

$$x^2 - 2xy + y^2 = 9. \quad (7)$$

Taking square root of (7),

$$x - y = \pm 3.$$

But (1) is $x + y = 5.$

Adding and subtracting,

$$2x = 8, \text{ or } 2; \qquad 2y = 2, \text{ or } 8.$$

Whence, $x = 4, \text{ or } 1; \qquad y = 1, \text{ or } 4.$

IV. Required to solve the equations,

$$\left.\begin{array}{rl} x^2 + y^2 = 52, & \\ x + y + xy = 34. & \end{array}\right\} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

Adding twice (2) to 1,

$$x^2 + 2xy + y^2 + 2x + 2y = 120. \qquad (3)$$

Or, $$(x + y)^2 + 2(x + y) = 120.$$

Completing square,

$$(x + y)^2 + 2(x + y) + 1 = 121.$$

Extracting square root,

$$x + y + 1 = \pm 11.$$

Whence, $$x + y = 10, \text{ or } -12. \qquad (4)$$

Squaring, $$x^2 + 2xy + y^2 = 100, \text{ or } 144. \qquad (5)$$

Subtracting (5) from twice (1),

$$x^2 - 2xy + y^2 = 4, \text{ or } -40.$$

Extracting square root, $x - y = \pm 2, \text{ or } \pm 2\sqrt{-10}.$

But (4) is $x + y = 10, \text{ or } -12.$

Whence, $2x = 12,\ 8,\ \text{or}\ -12 \pm 2\sqrt{-10}.$

And $2y = 8,\ 12,\ \text{or}\ -12 \mp 2\sqrt{-10}.$

From which $x = 6,\ 4,\ \text{or}\ -6 \pm \sqrt{-10}.$

And $y = 4,\ 6,\ \text{or}\ -6 \mp \sqrt{-10}.$

By substituting $u + v$ for x and $u - v$ for y, equations like the foregoing may often be elegantly solved.

V. Required to solve the equations,

$$\left.\begin{matrix} x^3 + y^3 = 152, \\ x + y = 8. \end{matrix}\right\} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Let $x = u + v$, and $y = u - v$.

Then $x + y = (u + v) + (u - v) = 2u.$

And $x^3 + y^3 = (u + v)^3 + (u - v)^3 = 2u^3 + 6uv^2.$

Hence, $2u = 8,$ and $u = 4.$

And $$2u^3 + 6uv^2 = 152. \qquad (3)$$

Substituting 4 for u in (3),

$$128 + 24v^2 = 152. \qquad (4)$$

Whence, $v^2 = 1,$

and $v = \pm 1.$

But $x = u + v = 4 \pm 1 = 5,\ \text{or}\ 3.$

And $y = u - v = 4 - (\pm 1) = 3,\ \text{or}\ 5.$

VI. Required to solve the equations

$$\left.\begin{aligned} x - y &= \tfrac{1}{4}xy, \\ x^2 + y^2 &= \tfrac{5}{8}xy. \end{aligned}\right\} \quad \begin{matrix}(1)\\(2)\end{matrix}$$

Let $x = u + v$, and $y = u - v$.

Then (1) becomes,

$$(u + v) - (u - v) = \tfrac{1}{4}(u + v)(u - v).$$

Or, $$2v = \tfrac{1}{4}(u^2 - v^2). \quad (3)$$

Also, (2) becomes,

$$(u + v)^2 + (u - v)^2 = \tfrac{5}{8}(u + v)(u - v).$$

Or, $$2u^2 + 2v^2 = \tfrac{5}{8}(u^2 - v^2). \quad (4)$$

From (4), $$\tfrac{1}{2}u^2 = \tfrac{1}{8}v^2.$$

Or, $$u^2 = 9v^2. \quad (5)$$

Substituting $9v^2$ for u^2 in (3),

$$2v = \tfrac{1}{4}(9v^2 - v^2).$$

Whence, $$2v = 2v^2,$$

and $$v = 1.$$

Substituting 1 for v in (5),

$$u = \pm 3.$$

But $x = u + v = \pm 3 + 1 = 4$, or -2.

And $y = u - v = \pm 3 - 1 = 2$, or -4.

Devices for the solution of simultaneous quadratic equations are numerous, but there are few general methods. The student's success will depend largely on his ingenuity. Some additional exercises will be found in the latter part of the book.

EXERCISES.

344. ***Find the roots of the following equations:***

1. $x + y = 20$; $x^2 + y^2 = 202$.
2. $x^2 + y^2 = 34$; $xy = 15$.
3. $x - y = 2$; $x^2 + y^2 = 34$.
4. $\sqrt{x} + \sqrt{y} = 6$; $x + y = 20$.
5. $x + y = xy$; $x^2 + y^2 = 5x^2y^2$.
6. $x + y = 5$; $x^3 + y^3 = 35$.
7. $x - y = 1$; $x^3 - y^3 = 37$.
8. $x^2 + y^2 = 81 - 2xy$; $x^2 + y^2 = 9 + 2xy$.
9. $xy + y^2 = 2$; $x^2 + xy = 12$.
10. $x^2 + y^2 - (x + y) = 18$; $xy + x + y = 19$.
11. $x + \sqrt{xy} + y = 14$; $x^2 + xy + y^2 = 84$.
12. $x^2 + y^2 = 2xy + 1$; $x^2 + xy + y^2 = 37$.
13. $x^2 - xy + y^2 = 7$; $x^4 + x^2y^2 + y^4 = 133$.
14. $x^2y + xy^2 = 20$; $x + y = \frac{4}{5}xy$.
15. $x^2 + y^2 = 20$; $(x + y)^2 + (x - y)^2 = \frac{4}{5}(x^2 - y^2)$.
16. $x + y = 72$; $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 6$.
17. $x + y - \sqrt{xy} = 7$; $x^2 + y^2 + xy = 133$.
18. $x^2 + y^2 = 13$; $2xy - (x + y) = 7$.
19. $xy(x + y) = 30$; $x^3 + y^3 = 35$.
20. $4(x + y) = 3xy$; $x + y + x^2 + y^2 = 26$.

MISCELLANEOUS EXERCISES.

345. *1.* $4(x^2 - y^2) = 15$; $2(x - y) = 3$.

2. $x^2 - y^2 = 102$; $x - y = 3$.

3. $4x^2 - 9y^2 = 76$; $2x - 3y = 2$.

4. $x + y = 74$; $x^2 + y^2 = 2740$.

5. $x - y = 4$; $x^2 + y^2 = 1066$.

6. $x + y = 7xy$; $12xy = 1$.

7. $x + y = 12$; $xy = 20$.

8. $x - y = 1$; $xy = 240$.

9. $x^2 + xy + 2y^2 = 44$; $2x^2 - xy + y^2 = 16$.

10. $2x^2 + 3xy + y^2 = 70$; $6x^2 + xy - y^2 = 50$.

11. $x^2 + xy + y^2 = 37$; $x^4 + x^2y^2 + y^4 = 481$.

12. $x^2 + y^2 - 1 = 2xy$; $xy(xy + 1) = 6$.

13. $x^2 + 3xy = 54$; $xy + 4y^2 = 115$.

14. $x^2 + xy - 8x = 3$; $y^2 + xy - 8y = 6$.

15. $x - y = 1$; $x^2 - xy + y^2 = 21$.

16. $5(x^2 - y^2) = 4(x^2 + y^2)$; $x + y = 8$.

17. $4x + 9y = 30$; $2x^2 - xy = 3y^2$.

18. $(x - 6)^2 + (y - 5)^2 + 2xy = 101$; $2x = 3y - 2$.

19. $4x^2 + 2xy + \frac{y^2}{4} + \frac{5(4x + y)}{12} = 41$; $4x = y + 4$.

20. $3x + 2y = 5xy$; $15x - 4y = 4$.

PROBLEMS LEADING TO SIMULTANEOUS QUADRATIC EQUATIONS.

346. I. Find two numbers whose sum is 38 and their product 261.

Solution.

Let x and y represent the numbers.

Then $$x + y = 38, \tag{1}$$

and $$xy = 261. \tag{2}$$

Squaring (1),

$$x^2 + 2xy + y^2 = 1444. \tag{3}$$

Multiplying (2) by 4, $$4xy = 1044. \tag{4}$$

Subtracting (4) from (3),

$$x^2 - 2xy + y^2 = 400. \tag{5}$$

Extracting square root of (5),

$$x - y = \pm 20. \tag{6}$$

But $$x + y = 38. \tag{1}$$

Whence, $2x = 58$, or 18, and $2y = 18$, or 58.

From which, $x = 29$, or 9; $y = 9$, or 29.

1. The sum of two numbers is 20 and their product is 99. What are the numbers?

2. The perimeter of a rectangular field is 104 rods and its area is 3 acres. Find its length and its breadth.

3. The perimeter, or sum of the four sides of a rectangle, is 112 rods, and its area is 720 square rods. What are the length and breadth of the rectangle?

II. Find two numbers whose difference is 6 and their product 391.

Solution.

Let x and y represent the numbers, x being the greater.

Then $$x - y = 6, \quad (1)$$

and $$xy = 391. \quad (2)$$

Squaring (1), $$x^2 - 2xy + y^2 = 36. \quad (3)$$

Multiplying (2) by 4, $$4xy = 1564. \quad (4)$$

Adding (3) and (4),

$$x^2 + 2xy + y^2 = 1600. \quad (5)$$

Extracting square root of (5),

$$x + y = \pm 40. \quad (6)$$

But $$x - y = 6. \quad (1)$$

Whence, $$x = 23, \text{ or } 17;$$

$$y = 17, \text{ or } 23.$$

4. The difference between two numbers is 11 and their product is 276. What are the numbers?

5. The difference between the length and the side of a rectangular field is 12 rods, and its area is 18 acres. What are its dimensions?

III. The sum of two numbers is 11, and the sum of their cubes is 407. What are the numbers?

Solution.

Let x and y represent the numbers, x being the greater.

Then $$x + y = 11, \quad (1)$$

and $$x^3 + y^3 = 407. \quad (2)$$

Dividing (2) by (1),

$$x^2 - xy + y^2 = 37. \quad (3)$$

Squaring (1), $x^2 + 2xy + y^2 = 121.$ (4)

Subtracting (3) from (4), $3xy = 84.$

Whence, $xy = 28.$ (5)

Subtracting (5) from (3),

$$x^2 - 2xy + y^2 = 9. \quad (6)$$

Extracting square root of (6),

$$x - y = \pm 3. \quad (7)$$

But $x + y = 11.$ (1)

Whence, $x = 7,$ or $4;$

$y = 4,$ or $7.$

6. The sum of two numbers is 14, and the sum of their cubes 854. What are the numbers?

7. Two cubical blocks of marble, when placed side by side, have a united length of twelve feet, and the sum of their volumes is 468 cubic feet. What are their dimensions?

IV. Two persons, A and B, depart from the same place, and travel in the same direction; A starts 2 hours before B, and after traveling 30 miles, B overtakes A; but had each of them traveled half a mile more per hour, B would have traveled 42 miles before overtaking A. At what rate did they travel?

Solution.

Let $x =$ A's rate per hour, and $y =$ B's rate per hour.

Then, $$\frac{30}{x} = \frac{30}{y} + 2. \quad (1)$$

And $$\frac{42}{x + \frac{1}{2}} = \frac{42}{y + \frac{1}{2}} + 2. \quad (2)$$

Clearing (1) and (2) of fractions,

$$30y = 30x + 2xy; \tag{3}$$

and $$42y + 21 = 42x + 21 + 2xy + x + y + \tfrac{1}{2}. \tag{4}$$

Subtracting (3) from (4),

$$11y = 13x + \tfrac{1}{2}.$$

Whence, $$y = \frac{26x + 1}{22}. \tag{5}$$

Substituting (5) in (3), clearing of fractions, and reducing,

$$26x^2 - 59x = 15. \tag{6}$$

Whence, $$x = 2\tfrac{1}{2}, \text{ or } -\tfrac{3}{13}.$$

Substituting $2\frac{1}{2}$ for x in (5),

$$y = 3.$$

8. A starts in pursuit of B, who has 3 hours' start, and overtakes him after traveling 36 miles. If the hourly rate of each had been $\frac{1}{2}$ of a mile more, A would have overtaken B at the end of $43\frac{1}{3}$ miles. Find their rates per hour.

9. The fore wheels of a carriage make 2 revolutions more than the hind wheels in going 90 yards; but if the circumference of each wheel is increased 3 feet, the carriage must pass over 132 yards in order that the fore wheels may make 2 revolutions more than the hind wheels. What is the circumference of each wheel?

10. A and B start at the same time, from two different points, and travel towards each other; when they meet on

the road, it appears that A has traveled 30 miles more than B. It also appears that it will take A 4 days to travel the road that B has come, and B 9 days to travel the road that A has come. Find the distance of A from B when they set out.

11. The area of a rectangular field is 1575 square rods; and if the length and breadth were each lessened 5 rods, its area would be 1200 square rods. What are the length and breadth?

12. If a certain number, consisting of two places, be divided by the product of its digits the quotient will be 2, and if 27 be added to it, the digits will be in an inverted order; required the number.

13. A man rents a farm for $420. He cultivates 4 acres himself, and lets the rest at $2½ per acre more than he pays for it. For this he receives the entire amount he is to pay for the farm. Find the number of acres and the rent per acre?

14. The fore wheel of a wagon makes 6 revolutions more than the hind wheel in going 120 yards; but if the periphery of each wheel be increased 1 yard, the fore wheel will make only four revolutions more than the hind wheel in going the same distance. What is the circumference of each wheel?

15. A grocer sold 80 lb. of mace and 100 lb. of cloves for $65; but he sold 60 lb. more of cloves for $20 than he did of mace for $10. What was the price of a pound of each?

16. A and B have each a small field in the shape of an exact square, and it requires 200 rods of fence to enclose both. The contents of these fields are 1300 square rods. What is the value of each, at $2.25 per square rod?

17. What two numbers are those whose difference multiplied by the difference of their squares is 32, and whose sum multiplied by the sum of their squares is 272?

18. The difference of two numbers is 2, and the square of their quotient added to four times their quotient is $9\frac{1}{4}$. What are the numbers?

19. There are two numbers whose sum multiplied by the less is equal to 36, but whose sum multiplied by the greater is equal to 45. What are the numbers?

20. Find two numbers such that their product added to their sum shall be 47, and their sum taken from the sum of their squares shall leave 62.

21. Two farmers sold their farms, which together contained 140 acres, and each received the same sum. If the first had received the same price per acre as the second, his farm would have brought \$1800. If the second had sold his farm at the same price per acre as the first, he would have received \$3200. How many acres in each farm?

22. A speculator bought as many sheep as cost him \$136. He reserved 22 of them, and sold the remainder for \$2 more than what they all cost him, gaining \$1 a head on them. How many sheep did he buy, and what was the price of each?

23. A speculator places \$12000 out at interest in two different sums. The rate per cent. is equal to the quotient arising from dividing the larger by the smaller, and the interest on the larger for one year is \$210 more than the smaller principal. What is each sum, and the rate per cent.?

24. A drover sold 60 oxen and 80 sheep for \$5300; he sold 42 more of the latter for \$450 than he did of the former for \$225. Find the price per head of each kind.

25. I invest the sum of \$2250 in two portions at different rates of interest, and they yield equal returns. The first portion at the second rate and the second portion at the first rate would yield an annual interest of \$40 and \$62.50 respectively. Find the rates.

RATIO.

347. Ratio is the relation with respect to magnitude that one quantity bears to another of the same kind.

348. The **terms** of a ratio are the quantities compared. The first term is called the **antecedent,** and the second the **consequent.** The terms of a ratio together form a **couplet.**

349. The ratio of one quantity to another is indicated by a colon; so used, the colon is a substitute for the sign of division. A ratio may also be written in the form of a fraction.

Thus, the ratio of 6 to 2 is written $6 \div 2$, or $\frac{6}{2}$.

350. The **value of a ratio** is equal to that of the fraction whose numerator is the antecedent of the ratio and its denominator the consequent.

Thus, the value of $6 : 2$ is $\frac{6}{2}$, or 3; of $8 : 3$ is $\frac{8}{3}$, or $2\frac{2}{3}$; of $a : b$ is $\frac{a}{b}$.

351. A **simple ratio** is a ratio consisting of two terms.

Thus, $4 : 8$ and $a : b$ are simple ratios.

352. A **compound ratio** is a combination of two or more simple ratios. It is compounded by taking the product of the corresponding terms of the simple ratios.

Thus, $\left.\begin{matrix} a:b \\ c:d \\ x:y \end{matrix}\right\}$ is a compound ratio.

Compounded, it is $acx : bdy$.

353. An **inverse** or **reciprocal ratio** is the ratio of the reciprocals of the two quantities compared.

Thus, the inverse ratio of a to b is $\frac{1}{a} : \frac{1}{b}$.

NOTE.—To distinguish $a : b$ from $\frac{1}{a} : \frac{1}{b}$, the former is called the **direct**, and the latter the **inverse** or **reciprocal ratio** of a to b.

354. With respect to any two quantities, as a and b,

$a^2 : b^2$	is called the	**duplicate ratio**	of	a	and b.
$a^3 : b^3$	" "	**triplicate ratio**	"	"	
$\sqrt{a} : \sqrt{b}$	" "	**subduplicate ratio**	"	"	
$\sqrt[3]{a} : \sqrt[3]{b}$	" "	**subtriplicate ratio**	"	"	

355. Any ratio may be written in fractional form. Hence,

1. *If the antecedent of a ratio is multiplied or the consequent divided, the value of the ratio is multiplied.*

2. *If the antecedent of a ratio is divided or the consequent multiplied, the value of the ratio is divided.*

3. *If both terms of a ratio be multiplied or divided by the same number, the value of the ratio is not changed.*

4. *A ratio is in the simplest form when its terms have no common factor.*

356. Two or more ratios may be compared with respect to magnitude by writing them in fractional form, reducing to a common denominator, and comparing the numerators.

Thus, let it be required to compare 2 : 3, 3 : 4, and 5 : 6.

$$\left.\begin{matrix}2:3\\3:4\\5:6\end{matrix}\right\} = (\tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{5}{6}) = (\tfrac{8}{12},\ \tfrac{9}{12},\ \tfrac{10}{12}).$$

Hence, the first ratio is the least and the last the greatest.

EXERCISES.

357. *1.* Reduce to their simplest form: 8 : 12, 15 : 25, 75 : 100, $\sqrt{144} : \sqrt{64}$, $\sqrt[3]{1000} : \sqrt{225}$.

2. Reduce to simplest form: $a^2b : ab^2$, $x^2 + 7x + 12 : x^2 + 4x$, $(x - y)^2 : x^2 - y^2$, $(x^2 - y^2)^2 : x^4 - y^4$.

3. What is the value of $6a^2b^2 : 3ab^3$? Of $\frac{2}{3} : \frac{4}{9}$? Of $\frac{5}{6} : \frac{1}{3}$?

4. What is the value of $\dfrac{(a-x)^2}{a^2+x^2} : a - x$?

Of $\dfrac{4c^2m}{3b^3y} : \dfrac{2cm}{3b^4y^2}$?

5. Simplify $\dfrac{x^2 - y^2}{x^3 + y^3} : \dfrac{x - y}{x^2 - xy + y^2}$.

Also $\dfrac{x^2 + 7x + 10}{x^2 + 3x - 10} : \dfrac{x^2 + 5x + 6}{x^2 - 5x + 6}$.

6. Compound the following ratios and simplify the result: 3 : 5, 16 : 21, 30 : 24.

7. Compound the duplicate ratio of 3 and 4 with the triplicate ratio of 3 and 2.

8. Two numbers are in the ratio of 3 to 4, and if 4 be added to each, they are in the ratio of 5 to 6. What are the numbers?

9. Arrange in the order of magnitude, $15 : 20$, $20 : 25$, $\sqrt{25} : \sqrt{64}$, $\sqrt[3]{125} : \sqrt[3]{216}$.

10. Find two numbers in the ratio of 3 to 5, whose sum is to the sum of their squares as $2 : 17$.

11. Which is greater, the inverse or the direct ratio of $\frac{1}{4}$ to $\frac{1}{5}$?

12. If 3 gold and 10 silver coins are worth as much as 2 gold and 20 silver coins, find the ratio of the value of a gold to that of a silver coin.

13. If 3 bushels of wheat and 11 bushels of corn are worth as much as 7 bushels of wheat and 5 bushels of corn, find the ratio of the value of a bushel of wheat to a bushel of corn.

14. A rectangular field containing 15 acres is in length to breadth as 3 to 2. What are its dimensions in rods?

15. Compound the ratios,

$$x + y : a, \quad x - y : b, \quad b^2 : \frac{x^2 - y^2}{a}.$$

16. Compound the ratios,

$$x^2 - 3x + 2 : x^2 - 5x + 6,$$

$$x^2 + 3x - 10 : x^2 + 3x - 4,$$

$$x^2 + x - 12 : x^2 + 7x + 10.$$

PROPORTION.

358. Proportion is an expression of the equality of two ratios. The equality is indicated either by a double colon or by the sign of equality.

Thus, $3 : 6 = 4 : 8$, or $3 : 6 :: 4 : 8$, is a proportion. It expresses the fact that $\frac{3}{6} = \frac{4}{8}$, and is read, *3 is to 6 as 4 is to 8.*

359. The **extremes** of a proportion are the first and fourth terms. The **means** are the second and third terms.

Thus, in the proportion $a : b :: c : d$, the extremes are a and d, and the means b and c.

360. A **mean proportional** between two quantities is a quantity to which either of the two bears the same ratio that the mean does to the other.

Thus, in the proportion $a : b :: b : c$, b is a mean proportional between a and c.

361. A proportion is taken by **inversion** when the terms of each ratio are written in inverse order.

Thus, $a : b :: c : d$ taken by inversion gives $b : a :: d : c$.

362. A proportion is taken by **alternation** when the means are made to change places, or the extremes.

Thus, $a : b :: c : d$ becomes by **alternation** either $a : c :: b : d$, or $d : b :: c : a$.

363. A proportion is taken by **composition** when the sum of the terms of each ratio is compared with either term of that ratio, the same order being observed in both ratios; or when the sum of the antecedents and the sum of the consequents are compared with either antecedent and its consequent.

Thus, $a : b :: c : d$ taken by composition gives $a + b : a :: c + d : c$, or $a + b : b :: c + d : d$, or $a + c : b + d :: a : b$, or $a + c : b + d :: c : d$.

364. A proportion is taken by **division** when in each ratio the antecedent minus the consequent, or the consequent minus the antecedent, is compared with either term in that ratio, the same order being observed in each ratio.

Thus, $a : b :: c : d$, by division becomes $a - b : a :: c - d : c$, or $a - b : b :: c - d : d$, or $b - a : a :: d - c : c$, or $b - a : b :: d - c : d$.

PRINCIPLES OF PROPORTION.

365. I. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

Thus, if $a : b :: c : d$, then $ad = bc$.

From the proportion, $\frac{a}{b} = \frac{c}{d}$.

Clearing of fractions, $ad = bc$.

366. II. *The square of a mean proportional equals the product of its extremes, and hence a mean proportional itself equals the square root of the product of its extremes.*

Thus, if $a : b :: b : c$, then $ac = b^2$.

For, $\frac{a}{b} = \frac{b}{c}$, which cleared of fractions gives $ac = b^2$.

Hence, $$b = \sqrt{ac}.$$

367. III. *Either extreme of a proportion equals the product of the means divided by the other extreme; and, in like manner, either mean equals the product of the extremes divided by the other mean.*

Thus, if $a : b :: c : d$, then $ad = bc$.

From this equation, $a = \frac{bc}{d}$, $d = \frac{bc}{a}$, $b = \frac{ad}{c}$, $c = \frac{ad}{b}$.

368. IV. *If the product of two quantities is equal to the product of two others, the factors of either product may be made the extremes, and the factors of the other product the means of a proportion.*

Thus, let $$ad = bc.$$

Dividing by bd, $$\frac{ad}{bd} = \frac{bc}{bd},$$

or $$\frac{a}{b} = \frac{c}{d}.$$

That is, $$a : b :: c : d.$$

369. V. *If four quantities are in proportion, they are in proportion by* **inversion.**

Thus, if $a : b :: c : d$, then $b : a :: d : c$.

From the proportion, $\frac{a}{b} = \frac{c}{d}$.

Hence, $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$;

that is, $\frac{b}{a} = \frac{d}{c}$.

From which, $b : a :: d : c$.

370. VI. *If four quantities are in proportion, they are in proportion by* **alternation.**

Thus, if $a : b :: c : d$, then $a : c :: b : d$.

From the proportion, $\frac{a}{b} = \frac{c}{d}$.

Multiplying by $\frac{b}{c}$, $\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$,

or $\frac{a}{c} = \frac{b}{d}$.

Whence, $a : c :: b : d$.

371. VII. *If four quantities are in proportion, they are in proportion by* **composition.**

Thus, if $a : b :: c : d$, then $a + b : b :: c + d : d$, and $a + b : a :: c + d : c$.

From the proportion, $\frac{a}{b} = \frac{c}{d}$.

Hence, $\frac{a}{b} + 1 = \frac{c}{d} + 1,$

or $\frac{a+b}{b} = \frac{c+d}{d}.$

Whence, $a + b : b :: c + d : d.$

Again, from the proportion taken by inversion,

$$b : a :: d : c.$$

Whence, $\frac{b}{a} = \frac{d}{c}.$

Hence, $1 + \frac{b}{a} = 1 + \frac{d}{c},$

or $\frac{a+b}{a} = \frac{c+d}{c}.$

That is, $a + b : a :: c + d : c.$

372. VIII. *If four quantities are in proportion, they are in proportion by* **division.**

Thus, if $a : b :: c : d$, then $a - b : b$ or $a :: c - d : d$ or c.

From the proportion, $\frac{a}{b} = \frac{c}{d}.$

Hence, $\frac{a}{b} - 1 = \frac{c}{d} - 1,$

or $\frac{a-b}{b} = \frac{c-d}{d}.$

Whence, $a - b : b :: c - d : d.$

Again, from the proportion taken by inversion,

$$b : a :: d : c.$$

Whence, $$\frac{b}{a} = \frac{d}{c}.$$

Hence, $$1 - \frac{b}{a} = 1 - \frac{d}{c},$$

or $$\frac{a - b}{a} = \frac{c - d}{c}.$$

That is, $$a - b : a :: c - d : c.$$

373. IX. *If four quantities are in proportion, they are in proportion by* **composition and division.**

Thus, if $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$.

By Prin. VII., $$\frac{a + b}{b} = \frac{c + d}{d}.$$

By Prin. VIII., $$\frac{a - b}{b} = \frac{c - d}{d}.$$

Hence, $$\frac{a + b}{b} \div \frac{a - b}{b} = \frac{c + d}{d} \div \frac{c - d}{d}.$$

That is, $$\frac{a + b}{a - b} = \frac{c + d}{c - d}.$$

Whence, $$a + b : a - b :: c + d : c - d.$$

374. X. *When any number of quantities are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Thus, if $a : b :: c : d :: e : f$, etc., then

$$a : b :: a + c + e, \text{ etc.} : b + d + f, \text{ etc.}$$

For, by Prin. I., $\left.\begin{matrix} ad = bc, \\ af = be. \end{matrix}\right\}$

And $ab = ba$, an identical eq.

Whence, by addition, $ab + ad + af = ba + bc + be.$

Factoring, $a(b + d + f) = b(a + c + e).$

Hence, by Prin. IV., $a : b :: a + c + e : b + d + f.$

375. XI. *The products of corresponding terms of two proportions are proportional.*

Thus, if $\left.\begin{matrix} a : b :: c : d \\ m : n :: r : s \end{matrix}\right\}$, then $am : bn :: cr : ds.$

From the proportions, $\frac{a}{b} = \frac{c}{d},$

and $\frac{m}{n} = \frac{r}{s}.$

Whence, $\frac{a}{b} \times \frac{m}{n} = \frac{c}{d} \times \frac{r}{s},$

or $\frac{am}{bn} = \frac{cr}{ds}.$

That is, $am : bn :: cr : ds.$

376. XII. *Like powers and like roots of proportional quantities are proportional.*

Thus, if $a : b :: c : d$, then

$$\left\{\begin{array}{l} a^n : b^n :: c^n : d^n, \\ \sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}. \end{array}\right\}$$

From the proportion, $\frac{a}{b} = \frac{c}{d}$.

Raising the equation to the n^{th} power; also extracting its n^{th} root,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}, \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}}.$$

Whence, $a^n : b^n :: c^n : d^n$,

and $\sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}$.

377. XIII. *If four quantities are proportional, the results will be proportional:*

1. *If the antecedents be multiplied or divided by the same number.*

2. *If the consequents be multiplied or divided by the same number.*

3. *If the terms of either ratio or of both ratios be multiplied or divided by the same or different numbers.*

Thus, let $a : b :: c : d$.

Then, by alternation, $a : c :: b : d$,

a proportion in which the first ratio contains the antecedents and the second ratio the consequents of the given proportion. But $\frac{a}{c}$ may be reduced to lower terms by division, or changed to higher terms by multiplication, and its value remain the same.

By means of this principle, a proportion may often be much simplified.

378. The application of the principles of proportion to the solution of equations expressing the equality of two fractions has already been shown. Principles VII., VIII., and IX. are most frequently used for this purpose.

EXERCISES.

379. *1.* When $\frac{1}{2}a - x : \frac{1}{2}a + x :: b - y : b + y$, show that $2x : y :: a : b$.

2. When $a : b :: x : y$, show that $ma + nb : ma - nb :: mx + ny : mx - ny$.

3. If $\frac{a^2 - x^2}{4a} = b$, show that $a - x : 2a :: 2b : a + x$.

4. If $(a + x)^2 : (a - x)^2 :: x + y : x - y$, show that $a : x :: \sqrt{2a - y} : \sqrt{y}$.

5. Divide 60 into two such parts that the product shall be to the sum of the squares as 2 : 5.

6. What is the value of x in $6x + a : 4x + b :: 3x - b : 2x - a$?

7. Given $x^3 - y^3 : (x - y)^3 :: 61 : 1$, and $x : 8 :: 40 : y$, to find the values of x and y.

8. Find the value of x in $\frac{3}{7} : \frac{3}{4} :: \frac{5}{6} : x$.

9. What number has the same ratio to 100 that 9 has to 20?

10. Make the following a correct proportion by changing each term in order: 22 : 10 :: 33 : 16.

11. Given $x + y : x :: 7 : 5$, and $xy + y^2 = 126$, to find the values of x and y.

12. Find two numbers, the greater of which shall be to the less as their sum is to 42, and as their difference is to 6.

13. Two squares whose sides differ by 10 rods have their areas to each other as 9 to 25. What are the sides of the squares?

14. There are two numbers whose product is 135, and the difference of their squares is to the square of their difference as 4 is to 1. What are the numbers?

15. A and B, with different sums, engage in business. A gains \$150, B loses \$50, after which A's money is to B's as 3 to 2; but had A lost \$50 and B gained \$100, then A's money would have been to B's as 5 to 9. What was each man's investment?

16. By the method of proportion, find the value of x in the equation,

$$\frac{\sqrt{m+x}+\sqrt{m-x}}{\sqrt{m+x}-\sqrt{m-x}} = n.$$

17. Find the value of x in the equation,

$$\frac{\sqrt[3]{x+1}+\sqrt[3]{x-1}}{\sqrt[3]{x+1}-\sqrt[3]{x-1}} = 2.$$

18. There is a number consisting of three digits, the first of which is to the second as the second is to the third; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to the number, the digits will be reversed. What is the number?

ARITHMETICAL PROGRESSION.

380. An **arithmetical progression** is a series of quantities, each of which is greater or less than the preceding quantity by a constant difference.

Thus, 1, 3, 5, 7, 9, etc., and 30, 20, 10, 0, — 10, etc., are *arithmetical progressions* or *series*.

381. The first of the foregoing is an **increasing** series, the second a **decreasing** series.

382. In every arithmetical progression five elements occur:

1. **The first term,** a.
2. **The last term,** l.
3. **The common difference,** d.
4. **The number of terms,** n.
5. **The sum of the terms,** S.

If any three of these elements are known, the other two may be found.

383. To find the last term of an arithmetical series, when the first term, the common difference, and the number of terms are given.

Remembering that in an increasing series d is positive, and in a decreasing series negative, the general expression

For the *second* term is $a + d$.
" *third* " $a + 2d$.
" *seventh* " $a + 6d$.
" *twentieth* " $a + 19d$.

Hence, for the n^{th} term, which is the *last* in a series of n terms,

$$l = a + (n - 1)\, d. \tag{1}$$

I. Required to find the 20th term of the series 1, 4, 7, 10, etc.

Solution.

Here $n = 20$, $d = 3$, $a = 1$, and the series is *increasing*. Substituting these values in formula (1), taking d as positive,

$$l = 1 + (20 - 1)\, 3 = 58.$$

II. Required to find the 25th term in the series 31, 25, 19, 13, etc.

Solution.

Here $n = 25$, $d = 6$, $a = 31$, and the series is *decreasing*. Substituting these values in (1) and taking d as negative,

$$l = 31 + (25 - 1) \times -6 = -113.$$

384. To find the sum S of an arithmetical series, when a, l, and n are given.

Taking d as the common difference, and writing the series in direct and in reverse order,

$$S = a + (a+d) + (a+2d) + \text{etc.}, + (l-2d) + (l-d) + l$$
$$S = l + (l-d) + (l-2d) + \text{etc.}, + (a+2d) + (a+d) + a$$

Adding,

$$2S = (a+l)+(a+l)+(a+l)+\text{etc.},+(a+l)+(a+l)+(a+l)$$

But since there are n terms in the series, the sum of the two series is equal to $a + l$ taken n times.

That is, $$2S = n\,(a + l).$$

Hence, $$S = \frac{n\,(a + l)}{2}, \quad \text{or} \quad \frac{n}{2}(a + l). \tag{2}$$

385. By substituting the value of l as given in formula (1) for l in formula (2), a value of S is found in terms of a, d, and n. Thus,

$$S = \frac{n}{2}(a + l)$$

$$= \frac{n}{2}[2a + (n - 1)\,d]. \tag{3}$$

III. Required to find the sum of an arithmetical series of 20 terms, the first term being 3, and the last term 79.

Solution.

Here $a = 3$, $l = 79$, and $n = 20$.

Hence, substituting in (2),

$$S = \tfrac{20}{2}\,(3 + 79) = 820.$$

IV. Required to find S when $a = 75$, $d = -4$, and $n = 30$.

Solution.

Substituting in (3),

$$S = \tfrac{30}{2}\,[150 + (30 - 1) \times -4] = 510.$$

386. To insert any number m of arithmetical means between a and l.

If m means are inserted, the series will consist of a and l, and m other terms; that is, n will equal $m + 2$.

Substituting this value of n in (1),

$$l = a + [(m + 2) - 1]\, d.$$

That is, $$l = a + (m + 1)\, d.$$

Whence, $$(m + 1)\, d = l - a,$$

and $$d = \frac{l - a}{m + 1}. \qquad (4)$$

By means of this formula the value of d may be found, after which the series is easily completed.

V. Required to insert five arithmetical means between 3 and 45.

Solution.

In this case, m is 5, a is 3, and l is 45.

Substituting in formula (4),

$$d = \frac{45 - 3}{5 + 1} = 7.$$

Hence, the series is 3, 10, 17, 24, 31, 38, 45.

387. In representing the terms of an arithmetical series by x and y,

1. When the number of terms is odd.

Represent the middle term by x, and make y the common difference.

Thus, a series of 5 terms becomes

$$x - 2y, \quad x - y, \quad x, \quad x + y, \quad x + 2y.$$

2. When the number of terms is even.

Make the adjacent terms farthest from the extremes $x - y$ and $x + y$, $2y$ being the common difference.

Thus, a series of 6 terms becomes

$$x - 5y, \quad x - 3y, \quad x - y, \quad x + y, \quad x + 3y, \quad x + 5y.$$

VI. In an arithmetical series of 4 terms, the sum of the squares of the extremes is 200, and the sum of the squares of the means is 136. What are the numbers?

Solution.

Let $x - 3y$, $x - y$, $x + y$, and $x + 3y$ represent the numbers.

Then $(x - 3y)^2 + (x + 3y)^2 = 200.$ (1)

And $(x - y)^2 + (x + y)^2 = 136.$ (2)

From (1), $2x^2 + 18y^2 = 200.$ (3)

From (2), $2x^2 + 2y^2 = 136.$ (4)

Subtracting (2) from (1), $16y^2 = 64.$

Whence, $y = \pm 2.$

And $x = \pm 8.$

Hence, the series is ± 2, ± 6, ± 10, ± 14.

EXERCISES.

388. *1.* Given $a = 2$, $d = 3$, and $n = 11$, to find l and S.

2. Given $a = 8$, $l = 203$, and $d = 5$, to find n and S.

3. Given $l = 1$, $S = 1717$, and $n = 34$, to find a and d.

4. Find the sum of the numbers 1, 2, 3, 4, etc., to 1000.

5. Given $a = 1$, $n = 23$, $S = 184$, to find d and l.

6. Insert 6 arithmetical means between 3 and 38.

7. Insert 3 arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$.

8. Given $a = -7$, $d = -7$, and $n = 101$, to find S.

9. If the first term of an arithmetical progression is 100, and the number of terms 21, what must the common difference be that the sum of the series may be 1260? What the last term?

10. Two persons, A and B, start from the same place together, and travel in the same direction. A goes 40 miles per day; B goes 20 miles the first day, and increases his rate of travel $\frac{1}{4}$ of a mile per day. How far will they be apart at the end of 40 days, and which will be in advance?

11. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on, how far will it fall during the t^{th} second?

12. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on, how far will it fall in t seconds?

13. A debt can be discharged in a year by paying \$1 the first week, \$3 the second, \$5 the third, and so on; required the last payment and the amount of the debt.

14. A person saves \$270 the first year, \$210 the second, and so on. In how many years will a person who saves every year \$180 have saved as much as he?

15. A board, $2\frac{1}{2}$ inches wide at the narrow end, and 10 feet long, increases in width $1\frac{1}{2}$ inches for every foot in length; what is the width of the wide end?

16. If 100 oranges are placed in a line, exactly 2 yards from each other, and the first 2 yards from a basket, what distance must a boy travel, starting from the basket, to gather them up singly, and return with each to the basket?

17. What four numbers are those in arithmetical progression, the sum of whose squares is 84, and their product 105?

18. The sum of five numbers in arithmetical progression is 35, and the sum of their squares 285; find the numbers.

19. What three numbers are those in arithmetical progression, the sum of whose squares is 1232, and the square of the mean greater than the product of the two extremes by 16?

20. Find four numbers in arithmetical progression such that the sum of the squares of the extremes is 4500, and the sum of the squares of the means is 4100.

21. The product of five numbers in arithmetical progression is 945, and their sum is 25. What are the numbers?

22. There is a number expressed by three digits in arithmetical progression. The division of the number by the sum of its digits gives 26, and when the number is increased by 198 its digits are reversed. What is the number?

23. Six heirs inherited an estate of $9000. Their shares formed an arithmetical progression, the common difference of which was $500. How much did each receive?

24. A body falling by its own weight, and not resisted by the air, would descend in the first second through a space of 16 feet and 1 inch; in the next second, through 3 times that space; in the third, through 5 times that space; in the fourth, through 7 times that space, etc. How far would it fall, at the same rate of increase, in a minute?

GEOMETRICAL PROGRESSION.

389. A **geometrical progression** is a series of quantities that increase or decrease by a constant multiplier called the **ratio.**

When the ratio is greater than 1, the series is **increasing**; when less than 1, the series is **decreasing.**

Thus, 1, 2, 4, 8, 16, etc., is an *increasing* geometrical progression or series, of which the ratio is 2.

And 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, etc., is a *decreasing* geometrical series, of which the ratio is $\frac{1}{3}$.

390. In every geometrical series five elements occur:

1. **The first term,** a.
2. **The last term,** l.
3. **The number of terms,** n.
4. **The ratio,** r.
5. **The sum of the series,** S.

If any three of these elements are known, the other two may be found.

391. To find the last term of a geometrical series, when the first term, the ratio, and the number of terms, are given.

From the definition of a geometrical series whose first term is a,

The *second* term is ar.
" *third* " ar^2.
" *ninth* " ar^8.
" *twentieth* " ar^{19}.

Hence, for the n^{th} term, which is the *last* in a series of n terms,

$$l = ar^{n-1}. \qquad (1)$$

I. Required to find the 7th term of the series, 2, 6, 18, 54, etc.

Solution.

Here $a = 2$, $r = 3$, $n = 7$.
Substituting these values in formula (1),

$$l = 2 \times 3^{7-1} = 2 \times 729 = 1458.$$

II. Required to find the 10th term in the series,

$$16, \quad -8, \quad 4, \quad -2, \quad \text{etc.}$$

Solution.

Here $a = 16$, $r = -\frac{1}{2}$, $n = 10$.
Substituting these values in formula (1),

$$l = 16 \times (-\tfrac{1}{2})^9 = 16 \times (-\tfrac{1}{512}) = -\tfrac{1}{32}.$$

392. To find the sum S of a geometrical series, when a, r, and n are given.

Indicating the sum of n terms of the series, and multiplying both members of the equation by r,

$$S = a + ar + ar^2 + \text{etc.}, + ar^{n-1}$$
$$Sr = \quad ar + ar^2 + \text{etc.}, + ar^{n-1} + ar^n$$

Subtracting the first equation from the second,

$$Sr - S = ar^n - a.$$

Whence,
$$S = \frac{a(r^n - 1)}{r - 1}. \qquad (2)$$

393. Taking the equation,

$$l = ar^{n-1},$$

multiplying both members by r,

$$lr = ar^n.$$

Substituting lr for ar^n in formula (2) gives,

$$S = \frac{lr - a}{r - 1}. \qquad (3)$$

When r is less than 1, formula (3) is most conveniently used when all the signs of the second member are changed. Thus,

$$S = \frac{a - lr}{1 - r}.$$

III. Required to find to 12 terms the sum of the series 1, 3, 9, 27, etc.

Solution.

Here $a = 1$, $r = 3$, $n = 12$.

Substituting these values in formula (2),

$$S = \frac{1(3^{12} - 1)}{3 - 1} = \frac{531440}{2} = 265720.$$

IV. Required to find the sum of a geometrical series of which the first term is 1536, the ratio $\frac{1}{2}$ and the last term 3.

Solution.

Substituting in formula (3) gives,

$$S = \frac{3 \times \frac{1}{2} - 1536}{\frac{1}{2} - 1} = \frac{-1534\frac{1}{2}}{-\frac{1}{2}} = 3069.$$

394. To find the sum of a decreasing geometrical series, when the number of terms is infinite.

For a decreasing series the formula is,

$$S = \frac{a - lr}{1 - r}.$$

In a decreasing series the successive terms approach zero more and more. It is evident, therefore, that such a series may be conceived as ending in zero. But if l becomes equal to zero, rl will be equal to zero, and the formula reduces to

$$S = \frac{a}{1 - r}. \qquad (4)$$

V. Required to find the sum of an infinite series of which the first term is 1, and the ratio $\frac{1}{3}$.

Solution.

Substituting these values in formula (4),

$$S = \frac{1}{1 - \frac{1}{3}} = 1\frac{1}{2}.$$

395. To insert m geometrical means between a and l.

The number of terms in the series will evidently be $m + 2$. That is, n in the formula $l = ar^{n-1}$ becomes $m + 2$.

Making the substitution, the formula is

$$l = ar^{m+1}.$$

Whence, $$r^{m+1} = \frac{l}{a}.$$

Hence, $$r = \sqrt[m+1]{\frac{l}{a}}. \qquad (5)$$

VI. Required to insert three geometrical means between 5 and 405.

Solution.

Substituting these values of a, l, and m in formula (5),

$$r = \sqrt[3+1]{\frac{405}{5}} = \sqrt[4]{81} = 3.$$

Hence, the series is 5, 15, 45, 135, 405.

396. There are many problems in geometrical progression to which the fundamental formulas do not apply directly. In solving them, much depends on the notation employed.

A convenient notation for a series of

Three terms is $\begin{cases} x,\ xy,\ xy^2, & \text{the ratio being } y. \\ x,\ \sqrt{xy},\ y, & \text{“ “ } \sqrt{\frac{y}{x}}. \end{cases}$

Four terms, $\begin{cases} x,\ xy,\ xy^2,\ xy^3, & \text{the ratio being } y. \\ \frac{x^2}{y},\ x,\ y,\ \frac{y^2}{x}, & \text{“ “ } \frac{y}{x}. \end{cases}$

Five terms, $\begin{cases} x,\ xy,\ xy^2,\ xy^3,\ xy^4, & \text{the ratio being } y. \\ \frac{x^3}{y},\ x^2,\ xy,\ y^2,\ \frac{y^3}{x}, & \text{“ “ } \frac{y}{x}. \end{cases}$

EXERCISES.

397. *1.* Find the fifth term of the series of which a is 3, and r is 5.

2. Given $a = 6561$, $n = 10$, $r = \frac{1}{3}$; to find l.

3. Find the sixth term of the series of which a is 2, and r is 3.

4. Given $a = 1$, $r = \frac{1}{2}$, $n = 8$; to find l.

5. What is the 13th term of the series 3, 6, 12, etc. ?

6. Given $a = 164$, $r = \frac{1}{2}$, $n = 13$; to find l.

7. Find S, when $a = 3$, $r = 2$, and $n = 8$.

8. Find S, when $a = 4$, $r = \frac{1}{3}$, and $n = 7$.

9. Insert 5 geometrical means between 5 and 3645.

10. Find the sum of the infinite series 1, $\frac{1}{2}$, $\frac{1}{4}$, etc.

11. Find the sum of $\frac{1}{2} + \frac{3}{8} + \frac{9}{32}$, etc., to infinity.

12. What is the value of $14 + 2 + \frac{2}{7}$, etc., to infinity ?

13. The sum of the first and third of four numbers in geometrical progression is 148, and of the second and fourth 888. What are the numbers ?

14. What three numbers in geometrical progression have their sum equal to 14, and the sum of their squares equal to 84 ?

15. Three numbers in geometrical progression have their sum equal to 52, and the sum of their extremes is to the mean as 10 to 3. What are the numbers ?

16. What is the distance passed through by a ball that falls 50 feet and each time rebounds half the distance ?

17. A gentleman divided $210 among three servants, so that their shares were in geometrical progression. How much did each receive if the second received $30 more than the last ?

PART II.

NUMERICAL VALUES.

398. Find the numerical value of the following expressions, when $a = 3$, $b = 5$, $c = 8$, $x = 10$, $y = 4$, $z = 6$, $u = 9$.

1. $\{(z + y) \div b\}\{(a + b) \div y\}$.

2. $\sqrt{a + b + c} + \sqrt{x(y + z)}$.

3. $\sqrt{x(y + z) - c(a + b)}$.

4. $\{(\sqrt[3]{a + b})(c + x)\} \div u$.

5. $x(a + y) \div b + u$.

6. $\left(\frac{z + u}{a} + \frac{x + y}{c}\right) \div \frac{y}{b}$.

7. $\sqrt{(\sqrt{u} + \sqrt{y})(c - a)}$.

8. $(b^2 - x) \div a - b$.

9. $\frac{y\sqrt{zu - b}}{b + c - z}$.

10. $\left(\frac{a}{b} + \frac{b}{z} + \frac{c}{u}\right)\frac{u}{x}$.

11. $\frac{[x + 2b \times ab]y - cxu}{z}$.

When $a = 4$, $b = 9$, $c = 5$, $d = 12$, $e = 1$, $m = 8$, $n = 25$, $x = 15$, $y = 10$, $z = 3$.

12. $\left(\frac{a}{m} + \frac{c}{y} + \frac{z}{b}\right)\frac{d}{a} - \frac{3n + c}{d + m}$.

13. $(y-a)\sqrt{bm-2a} \div \sqrt[5]{6\left(z^2+a^2+\frac{d+y}{2}\right)}$.

14. $\left[\frac{a+b+c}{9}+\frac{2m+\frac{1}{2}(n+x)}{x+z}\right] \div \frac{\frac{5}{6}cd}{b+e}$.

15. $\frac{z(a+b)+c(n+x)+d-e}{cm(a+e)+y\sqrt{n}}$.

When $a=1$, $b=2$, $c=4$, $d=6$, $e=2$, $f=0$.

16. $\frac{cd}{b}+\frac{4be}{3a}-\frac{cd}{24}$.

17. $abc+4bd+ec-fd$.

18. $\frac{b^2+c^2}{2c-3a}$.

19. $\sqrt{24d}-\sqrt[3]{2c}+\sqrt{2e}$.

20. c^4-4c^3+3c-6.

21. $a^2+b^2+c^2+d^2$.

22. $\frac{d^3-c^3}{d^2+cd+c^2}$.

23. $\sqrt{2bc}+\sqrt[3]{9cd}-\sqrt[3]{2c^2}$.

When $a=4$, $b=9$, $c=5$, $d=12$, $e=1$, $m=8$, $n=25$, $x=15$, $y=10$, $z=3$.

24. $\left\{c\sqrt{\frac{a}{n}}+z\sqrt[3]{\frac{m}{cn}}\right\} \div \left\{e\sqrt[4]{\frac{a+d}{nx+c^2y}}\right\}^{\frac{1}{2}}$.

25. $\left\{\frac{(2a-3b+4n)^{\frac{1}{2}}(abn)^{\frac{1}{2}}}{\left[(5y-3x)(m-2z)\sqrt{5d+8c}\right]^{\frac{1}{2}}}\right\}$.

26. $\left\{z\sqrt{xyz+c\sqrt[3]{(x+y)(a+e)}\sqrt{ad+e}}-y+3z-\sqrt{2ab+a\sqrt{cd-(m+z)}}\right\}^{\frac{1}{2}}$.

When $a = 3$, $b = 5$, $c = 7$, $d = 0$, $m = 2$, $n = 1$, $r = \frac{1}{2}$, $s = \frac{1}{3}$, $x = 4$, $y = 10$, $z = 6$.

27. $\{x\sqrt{\frac{1}{2}r}\,(\sqrt{c+m}) \div [\sqrt{10rsyz} - \sqrt[5]{y+z-ab}]^{\frac{1}{2}}\}^3$.

28. $\Big[x\sqrt[3]{(c+m)^{\frac{1}{2}}\,(y+z)^{\frac{1}{4}} + ac} + \sqrt{z\sqrt{5b} + \tfrac{1}{4}\,(y+m)} + cmn\Big]^{\frac{1}{2}}$.

29. $[(abc - brsz)^{\frac{1}{2}} - \{(c-a)\sqrt[3]{x+yz}\}^{\frac{1}{4}}\,\{2mn^2r^2x\sqrt{3as^2x}\}^{\frac{1}{2}} \times \{br\sqrt{y\,(a+c+2dn^2)}\} + (a+n)\sqrt{bc+n} + n]^{\frac{1}{2}}$.

ADDITION AND SUBTRACTION.

399. ***Find the sum of:***

1. $3a + b - c$, $9a - 6b - 10c + f$, $8a + 6b - 8c + 3f$, $4a - b + c + g$, and $6a - 3b - 8c$.

2. $6x^2 + 3x^3 + 2a - 7b$, $8x^2 - 8b + 2x^3$, $5a - 6x^3 - 4b$, and $-7a + 2b$.

3. $3am + m^2 + 8\sqrt{m+n}$, $5am - 9\sqrt{m+n}$, $-3m^2 + \sqrt{m+n}$, and $-8am + 2m^2 - \sqrt{m+n}$.

4. $4x^{\frac{1}{2}} + 2xy + y^2 + 4$, $-3x^{\frac{1}{2}} - 5xy - 2y^2 + 11$, and $3xy + 7y^2 + 3$.

5. $5a^2 + 2ab + 3b^2$, $5b^2 + b^3 - 2a^2$, and $-3a^2 - ab - 2b^3$.

6. $a^2 + 3b^2x + x^2$, $2b^2x - 4a^2 - 9x^2$, and $3a^2 + 8x^2 - 4b^2x + 3c^2$.

7. $a^m - b^n + 3x^p$, $2a^m - 3b^n - x^p$, and $a^m + 4b^n - x^p$.

8. $a^{-1} + 3b^{-2} - c^2 + d$, $-3a^{-1} + b^{-2} + 18c^2 - d$, and $4a^{-1} - 5b^{-2} + 3c^2$.

9. $5\sqrt{x} - 3\sqrt{y} + 5$, $2\sqrt{y} - x - 3$, and $2\sqrt{x} + \sqrt{y} + 8$.

10. $3a^2c^{\frac{1}{2}} - 2c^2a^{\frac{1}{2}} + a^{\frac{1}{2}}c^{\frac{1}{2}}$, $2a^2c^{\frac{1}{2}} + 3c^2a^{\frac{1}{2}} - 5a^{\frac{1}{2}}c^{\frac{1}{2}}$, and $a^2c^{\frac{1}{2}} - 5c^2a^{\frac{1}{2}} + 8a^{\frac{1}{2}}c^{\frac{1}{2}}$.

11. $5(a - x^2) + 3\sqrt{a - x} + 5$, $4(a - x^2) - 2\sqrt{a - x} + 8$, $2(a - x^2) - 8\sqrt{a - x} - 12$, and $-(a - x^2) + 2\sqrt{a - x} - 1$.

12. $a^m - 2b^n + 3c^p - 4d^q$, $23a^m + 16b^n - 13c^p + 12d^q$, $-14a^m + 15b^n - 17c^p + 19d^q$, $18a^m - b^n + c^p - d^q$, and $-13a^m + 5b^n - 3c^p + 11d^q$.

13. $a^3 - \frac{3}{4}a^2x + \frac{1}{3}ax^2 - \frac{5}{8}x^3$, $a^3 + \frac{1}{2}a^2x + \frac{1}{4}ax^2 + \frac{2}{5}x^3$, and $2a^3 - \frac{1}{4}a^2x + \frac{7}{12}ax^2 + \frac{1}{24}x^3$.

14. $a - \frac{1}{3}b + \frac{1}{4}c - \frac{1}{6}d$, $-\frac{1}{2}c + \frac{1}{3}a - \frac{1}{4}b + d$, $\frac{1}{4}d - \frac{1}{5}b + c - a$, $\frac{1}{6}a - \frac{2}{3}d + b - \frac{5}{6}c$, and $8a - 6b + 3c - 4d$.

15. From $-a^2x^2 + 4axy - 3aby^2$ take $4aby^2 - 5axy + 2a^2x^2$.

16. From $\sqrt{x^2 - y^2} + 4(x+y) - 3\sqrt{a + x}$ take $3(x+y) - 2\sqrt{x^2 - y^2} + 3\sqrt{a + x}$.

17. From $2a^2 + ax + x^3 - 12a^2x + 20ax^2 - 4x^3 + 6a^2x^3 - 10ax^3$ take $a^2 - 3ax + 2x^2 - 16a^2x + 12ax^2 - 12ax^3 - 4x^3 + 2a^2x^2$.

18. From $2x\sqrt{x+y} - 3axy + 2abc$ take $-17axy + 11abc - x\sqrt{x+y}$.

19. From $9ax - 13 + 20ab^3x - 4bcx^2$ take $3bcx^2 + 9ax - 6 + 3ab^3x$.

20. From $5a^4 - 7a^3b^2 - 3cd^2 + 7d$ take $-15a^3b^2 + 3a^4 - 3a^2 - 7cd^2$.

MULTIPLICATION.

400. ***Multiply:***

1. $4x^3 + 8x^2 + 16x + 32$ by $3x - 6$.

2. $a^2 - ab + ac + bc + b^2 + c^2$ by $a + b - c$.

3. $x^4 + 3x^3 + 5x^2 + 4x + 1$ by $x^3 - 3x^2 + 4x - 1$.

4. $x^5 + 3x^4 + 6x^3 + 10x^2 + 15x + 21$ by $x^3 - 3x^2 + 3x - 1$.

5. $3x^3 + 2x^2 - 5x$ by $4x^2 - 7x$.

6. $2x^3 + 4x^2 + 8x + 16$ by $3x - 6$.

7. $16a^3 - 24a^2b + 36ab^2 - 27b^3$ by $2a + 3b$.

8. $x^4 + 2x^3 + 3x^2 + 2x + 1$ by $x^2 - 2x + 1$.

9. $3x^{-2} - 5x^m y^{-n} + z^{-1}$ by $2x^{-4}y^m$.

10. $\frac{1}{3} + 2x + 3x^2$ by $1 - 6x + 9x^2$.

11. $-a^{-5}x^2 + 7a^{-1}x^3 + 8a^3x^4$ by $2a^{-3}x^3 - 3ax^4$.

12. $\frac{2}{3}x^3 - 5x^2 + \frac{1}{4}x + 9$ by $\frac{1}{2}x^2 - x + 3$.

13. $x^{\frac{1}{2}} - x^{\frac{1}{2}}y^{\frac{2}{3}} + x^{\frac{1}{2}}y^{\frac{4}{3}} - y^{\frac{2}{3}}$ by $x^{\frac{1}{2}} + y^{\frac{2}{3}}$.

14. $x^3 - x^2y^{-\frac{2}{3}} + xy^{-\frac{4}{3}} - y^{-2}$ by $x + y^{-\frac{2}{3}}$.

15. $\frac{3}{5}a^3b^2 - \frac{3}{4}a^2b^3 + 6ab^4$ by $\frac{1}{2}a^4 - 2a^3b$.

16. $\frac{5}{6}x + \frac{3}{4}y + 5z$ by $-\frac{3}{5}x + 5y$.

17. $x^{2n} - 2x^ny^n + y^{2n}$ by $x^n - y^n$.

18. $\frac{3}{5}a^2 - \frac{1}{6}a + 1$ by $\frac{1}{2}a^2 - \frac{3}{4}a + 6$.

19. $\frac{3}{2}x^3 - 5x^2 + \frac{1}{4}x + 9$ by $\frac{1}{2}x^2 - x + 3$.

20. $6x^4 - 3x^3y + 3x^2y^2 - 3xy^3 + \frac{3}{8}\frac{1}{6}y^4$ by $x^3 + \frac{1}{2}x^2y - \frac{1}{4}xy^2 + \frac{1}{8}y^3$.

DIVISION.

401. ***Divide:***

1. $a^5 + 32b^5$ by $a + 2b$.

2. $8m^{10} - 9m^9 + m$ by $m^2 - 2m + 1$.

3. $\frac{1}{3} - 6a^2 + 27a^4$ by $\frac{1}{3} + 2a + 3a^2$.

4. $\frac{1}{8}a^3 - \frac{1}{3}a^2b + \frac{1}{4}ab^2 - \frac{1}{2}b^3$ by $2a - \frac{1}{2}b$.

5. $x^2 + \frac{5}{6}x + \frac{1}{6}$ by $x + \frac{1}{3}$.

6. $x^4 - \frac{1}{4}a^2x^2 + a^3x - a^4$ by $x^2 + \frac{1}{2}ax - a^2$.

7. $a^2 + ab + b^2$ by $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$.

8. $x^{\frac{3}{4}} - a^3$ by $x^{\frac{1}{4}} - a^{\frac{1}{2}}x^{\frac{1}{8}} + a$.

9. $x^{2n} - y^{2n}$ by $x^n + y^n$.

10. $x^{4n} + x^{2n}y^{2n} + y^{4n}$ by $x^{2n} + x^ny^n + y^{2n}$.

11. $16a^3x^2 - y$ by $2a^{\frac{3}{4}}x^{\frac{1}{2}} + y^{\frac{1}{4}}$.

12. $m^6 - \frac{5}{21}m^3n^3 - \frac{2}{3}m^2n^4 + \frac{16}{189}n^6$ by $m^3 + m^2n + \frac{1}{3}mn^2 - \frac{8}{21}n^3$.

13. $48a^{\frac{11}{6}} + \frac{1}{12}a^{\frac{23}{6}} + \frac{1}{864}$ by $6a^{\frac{11}{6}} - 3a^{\frac{11}{12}} + \frac{1}{2}$.

14. $x^3 - 8x^{\frac{6}{5}} - x^{\frac{7}{4}} + 136x^{\frac{3}{4}} - 128$ by $x^{\frac{3}{5}} - 2x^{\frac{1}{5}} - x^{\frac{1}{4}} + 2$.

15. $24a^{12} - \frac{40}{9}a^{\frac{3}{5}}b^{-2} - \frac{43}{54}b^{-\frac{16}{7}}$ by $4a^3 - 4a^{\frac{3}{5}}b^{-\frac{2}{7}} - \frac{2}{3}b^{-\frac{4}{7}}$.

16. $12m^r - 11m^{\frac{2r}{5}}n^{\frac{4s}{15}} - n^{\frac{2s}{5}}$ by $4m^{\frac{2r}{5}} - 2m^{\frac{r}{5}}n^{\frac{2s}{15}} - 2n^{\frac{4s}{15}}$.

17. $96x^{\frac{11}{3}} - x^3 - \frac{3}{2}x^{\frac{4}{3}} - \frac{5}{12}x - \frac{1}{12}$ by $16x^{\frac{4}{3}} - 8x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{1}{3}} - \frac{3}{4}$.

FACTORING.

402. Arts. **122** and **124** show the method of factoring the sum or the difference of *equal odd powers* of two quantities. This method may be extended to cases where the terms of the binomial may be *expressed* as *equal odd powers*.

Thus,

$x^3 + y^6 = x^3 + (y^2)^3 = (x + y^2)(x^2 - xy^2 + y^4)$.

$x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

$x^{10} - y^{15} = (x^2)^5 - (y^3)^5 = (x^2 - y^3)(x^8 + x^6y^3 + x^4y^6 + x^2y^9 + y^{12})$.

$x^{\frac{3}{4}} - y^{\frac{6}{7}} = (x^{\frac{1}{4}})^3 - (y^{\frac{2}{7}})^3 = (x^{\frac{1}{4}} - y^{\frac{2}{7}})(x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{2}{7}} + y^{\frac{4}{7}})$.

403. *Factor the following:*

1. $a^6 + b^3$, $x^6 - y^9$, $x^9 + y^6$, $x^{12} - y^{15}$, $p^9 - q^{12}$.

2. $x^{10} + y^{10}$, $x^{12} + y^{12}$, $a^5 - b^{10}$, $m^5 + n^{10}$, $a^{18} + x^6$.

3. $x^{18} + y^{18}$, $a^{15} - n^{15}$, $p^{20} - q^{15}$, $a^{18} - y^{15}$, $x^{18} + y^{12}$.

4. $8x^6 + 27y^6$, $125x^{12} - 64y^{15}$, $216a^9 - 1000b^6$, $8m^{15} + 64n^9$.

5. $\frac{8}{125}x^6 + \frac{27}{64}y^9$, $\frac{8}{27}m^{\frac{3}{5}} - \frac{125}{216}n^{\frac{3}{5}}$, $\frac{343}{125}x^{\frac{6}{5}} + \frac{27}{64}y^{\frac{3}{4}}$.

6. $x^6y^9 + 8$, $32x^{10} - y^{15}$, $a^6b^{15} + x^{12}y^9$, $64x^6 + 729y^6$.

HIGHEST COMMON DIVISOR.

404. ***Find the H. C. D. of the following:***

1. $x^3 - 5x^2 - 17x + 21$ and $x^3 - 3x^2 - 25x - 21$.

2. $x^3 + 11x^2 + 31x + 21$ and $2x^3 + 17x^2 + 16x - 35$.

3. $x^3 - 6x^2 + 11x - 6$ and $x^3 + 4x^2 + x - 6$.

4. $x^3 - 19x^2 + 119x - 245$ and $3x^2 - 38x + 119$.

5. $x^3 - 8x^2 + 19x - 12$ and $4x^3 - 27x^2 + 58x - 39$.

6. $2x^3 + 9x^2 + 4x - 15$ and $4x^3 + 8x^2 + 3x + 20$.

7. $3x^4 - x^2 - 2$ and $2x^3 + 3x^2 - 2x - 3$.

8. $5x^3 - 18x^2 + 11x - 6$ and $x^4 - 9x^3 + 29x^2 - 39x + 18$.

9. $x^4 - 4x + 3$ and $35x^5 - 20x^2 + 15x - 30$.

10. $x^5 + 5x^3 + 6$ and $x^4 + 40x + 39$.

11. $4x^3 - 10x^2 + 6x - 15$ and $14x^4 - 35x^3 + 4x^2 - 12x + 5$.

12. $x^5 - 82xy^4 - 3y^5$ and $x^5 + 28x^2y^3 - 9y^5$.

13. $x^6 - 48x^2y^4 + 7y^6$ and $7x^6 - 48x^4y^2 + y^6$.

14. $a^5b - 5a^3bc^2 - 19a^2bc^3 - 49bc^5$ and $a^5 - 27a^2c^3 - 77ac^4 - 245c^5$.

15. $x^6 - 15x^2 - 24x - 10$ and $x^5 + 10x^2 + 15x + 6$.

16. $2x^6 - 9x^4 - 8x^3 - 1$ and $x^6 - 15x^2 - 24x - 10$.

LOWEST COMMON MULTIPLE.

405. ***Find the L. C. M. of the following:***

1. $x^2 + x - 30$, $x^2 + 11x + 30$, and $x^2 - x - 42$.

2. $7(x - y)$, $14(x^2 - y^2)$, and $21(x^3 - y^3)$.

3. $11(a^3b - ab^3)$, $33(a^4 - b^4)$, and $22(a^2 - b^2)$.

4. $2x^2 + 3x - 2$ and $x^2 + x - 2$.

5. $x^2 + 2x - 3$, $x^3 + x^2 - 14x - 24$, and $x^3 - 3x^2 - 6x + 8$.

6. $16x^2 - 20x - 6$ and $2x^3 - 5x^2 + 5x - 3$.

7. $x^3 - 7x - 6$, $x^3 + 7x^2 + 14x + 8$, and $x^3 + 3x^2 - 10x - 24$.

8. $x^3 - 9x^2 + 26x - 24$, $x^3 - 10x^2 + 31x - 30$, and $x^3 - 11x^2 + 38x - 40$.

9. $x^5 - 5x^3 + x^2 + 4x - 4$ and $x^4 + x^3 - 6x^2 - 4x + 8$.

10. $x^4 - 7x^3 - 7x^2 + 43x + 42$ and $x^4 - 9x^3 + 9x^2 + 41x - 42$.

11. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$, and $x^3 - 8x^2 + 19x - 12$.

12. $x^3 - 9x^2 + 26x - 24$, $x^3 - 10x^2 + 31x - 30$, and $x^3 - 11x^2 + 38x - 40$.

13. $x^4 - 5x^3 - 7x^2 + 29x + 30$ and $x^4 - x^3 - 7x^2 + x + 6$.

14. $x^3 - 2x^2 - 29x - 42$, $x^3 - 8x^2 + x + 42$, and $x^3 - 7x^2 - 9x + 63$.

FRACTIONS.

406. ***Reduce the following to their simplest forms:***

1. $\dfrac{a^2x^2y^2}{a^3xy + axy^3}$.

2. $\dfrac{axy + xy^2}{axy}$.

3. $\dfrac{11a^2 - 22ax}{33\,(a^2 - 4x^2)}$.

4. $\dfrac{9x^2y^3 - 15xy^4}{12x^2y^2 - 21xy^3}$.

5. $\dfrac{x^4 - b^4}{x^5 - b^2x^3}$.

6. $\dfrac{a^3 + 2a^2}{a^2 + 4a + 4}$.

7. $\dfrac{abc + 9bc - 5c^2}{2abdf + 18bdf - 10cdf}$.

8. $\dfrac{a^2 - ab - 2b^2}{a^2 - 3ab + 2b^2}$.

9. $\dfrac{x^3 + 3x^2 - 4}{x^3 - 1}$.

10. $\dfrac{3ax^2 - 13ax + 14a}{7x^3 - 17x^2 + 6x}$.

11. $\dfrac{2x^3 + 5x^2 - 3x}{x^3 + 2x^2 - 3x}$.

12. $\dfrac{x^4 + a^2x^2 + a^4}{x^4 + ax^3 - a^3x - a^4}$.

13. $\dfrac{15x^3 + 35x^2 + 3x + 7}{27x^4 + 63x^3 - 12x^2 - 28x}$.

14. $\dfrac{x^4 + (2b^2 - a^2)\,x^2 + b^4}{x^4 + 2ax^3 + a^2x^2 - b^4}$.

15. $\dfrac{a^5 + 3ab^4 - 4a^2b^3}{a^4 - a^3b - ab^3 + b^4}$.

16. $\dfrac{x^2 - (a - b)\,x - ab}{x^3 + bx^2 + ax + ab}$.

17. $\dfrac{x^3 + x - 12}{x^3 - 5x^2 + 7x - 3}$.

18. $\dfrac{x^3 + (1 + x)\,xy + y^2}{x^4 - y^2}$.

19. $\dfrac{x^4 - (3m + 5n^2)\,x^2 + 15mn^2}{x^4 - (3m - 5n^2)\,x^2 - 15mn^2}$.

20. $\dfrac{(ab - 1)^2 + (a + b - 2)\,(a + b - 2ab)}{(ab + 1)^2 - (a + b)^2}$.

407. ***Find the value of the following:***

1. $\dfrac{5x + 8}{6} - \dfrac{2x - 9}{3}$.

2. $\dfrac{7x^2 - 8xy}{9} - \dfrac{xy - 2x^2}{18}$.

3. $\dfrac{5a^2 + b}{5} + \dfrac{4a^2 + 2b^2}{5b}$.

4. $\dfrac{2x + 1}{3x} + \dfrac{4x + 2}{5x} + \dfrac{1}{7}$.

5. $\dfrac{4x + 2}{3} - \dfrac{2x - 3}{3x}$.

6. $\dfrac{x\,(a + x)}{a - x} - \dfrac{5ax - x^2}{x - a}$.

7. $\dfrac{2x^2 + 4xy + 2y^2}{2x^2 + 2xy} + \dfrac{x}{x - y}$.

8. $\dfrac{a}{a - x} + \dfrac{3a}{a + x} - \dfrac{2ax}{a^2 - x^2}$.

9. $\dfrac{x^2}{x^2 - 1} - \dfrac{x}{1 - x} + \dfrac{x}{x + 1}$.

10. $\dfrac{1}{a + x} + \dfrac{1}{a - x} + \dfrac{2a}{a^2 + x^2}$.

11. $\dfrac{3a - 4b}{7} - \dfrac{2a - b - c}{3} + \dfrac{15a - 4c}{12} - \dfrac{a + 64b}{84}$.

12. $\dfrac{5x^2+2}{x-1} - \dfrac{3x^2-5x-12}{1-x^2} - \dfrac{5x^2-2}{x+1}$.

13. $\dfrac{1}{4(1+y)} - \dfrac{1}{4(y-1)} + \dfrac{1}{2(1+y^2)}$.

14. $\dfrac{b-2a}{x-b} - \dfrac{a-2b}{x+b} - \dfrac{3x(a-b)}{b^2-x^2}$.

15. $\dfrac{1}{1+x} - \left\{\dfrac{6}{1-x} - \left(\dfrac{2}{1+2x} - \dfrac{16}{2x-1}\right)\right\}$.

16. $\dfrac{1}{x^2(x^2+y^2)} - \dfrac{1}{2x^2(y-x)} + \dfrac{1}{2x^2(x+y)}$.

17. $\dfrac{1-x+x^2}{1+x-x^2} + \dfrac{6x^3-4x^4}{1+x-x^2} - 4x^2 + 2x - 1$.

18. $\dfrac{3+2x}{2-x} - \dfrac{2-3x}{x+2} - \dfrac{x^2-16x}{x^2-4}$.

19. $\dfrac{x^2-yz}{(x+y)(x+z)} + \dfrac{y^2-xz}{(x+y)(y+z)} + \dfrac{z^2-xy}{(x+z)(y+z)}$.

20. $\dfrac{x^2-(y-z)^2}{(x+z)^2-y^2} + \dfrac{y^2-(x-z)^2}{(x+y)^2-z^2} + \dfrac{z^2-(x-y)^2}{(y+z)^2-x^2}$.

21. $\dfrac{x^2-9x+20}{x^2-6x} \times \dfrac{x^2-13x+42}{x^2-5x}$.

22. $\dfrac{x+1}{x-1} \times \dfrac{x^2+x-2}{x^2+x}$.

23. $\dfrac{3x^2-x}{5} \times \dfrac{10}{2x^2-4x}$.

24. $\dfrac{a^2-x^2}{a+b} \times \dfrac{a^2-b^2}{ax+x^2} \times \left(a + \dfrac{ax}{a-x}\right)$.

25. $\left(1 - \frac{a-b}{a+b}\right) \times \left(2 + \frac{2b}{a-b}\right)$.

26. $\frac{x(a-x)}{a^2+2ax+x^2} \times \frac{x(a+x)}{a^2-2ax+x^2}$.

27. $\frac{m^2-mn+n^2}{m^3-3mn(m-n)-n^3} \times \frac{m^2-n^2}{m^3+n^3}$.

28. $\left\{\frac{2x}{x^2+1} + \frac{2x}{x^2-1}\right\} \div \left\{\frac{x}{x^2+1} - \frac{x}{x^2-1}\right\}$.

29. $\frac{a+b+\frac{b^2}{a}}{a+b+\frac{a^2}{b}}$.

30. $\frac{a+\frac{b-a}{1+ab}}{1-\frac{a(b-a)}{1+ab}}$.

31. $\frac{\frac{a^3+b^3}{a^2-b^2}}{\frac{a^2-ab+b^2}{a-b}}$.

32. $\frac{\frac{2a+b}{a+b}-1}{1-\frac{b}{a+b}}$.

33. $\frac{\frac{1}{1+x}}{1-\frac{1}{1+x}} + \frac{\frac{1}{1+x}}{\frac{x}{1-x}} + \frac{\frac{1}{1-x}}{\frac{x}{1+x}}$.

34. $\frac{b}{b-rx^2} - \left[\frac{3b}{rx^2-b} + \frac{2brx^2}{r^2x^4-b^2}\right]$.

35. $\frac{a^2}{(x-a)(a-b)} + \frac{b^2}{(x-b)(b-a)}$.

36. $\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}$.

37. $\frac{1}{x(x-y)(x-z)} + \frac{1}{y(y-x)(y-z)} - \frac{1}{xyz}$.

38. $\dfrac{yz}{(x-z)(y-x)} + \dfrac{xz}{(x-y)(y-z)} + \dfrac{xy}{(z-y)(x-z)}$.

39. $\dfrac{x+y}{(x-5)(5-y)} - \dfrac{x+5}{(x-y)(y-5)} - \dfrac{y+5}{(x-y)(5-x)}$.

40. $\dfrac{(a-b)^2}{(a-b)-1} + \dfrac{(a-b)^2}{b-a-1} + \dfrac{1}{b+1-a} + \dfrac{1}{a-b+1}$.

41. $\dfrac{x-a}{(a+y)(y-x)} + \dfrac{x+y}{(a+x)(a+y)} + \dfrac{a-y}{(a+x)(y-x)}$.

42. $\dfrac{x^2-6}{x^2-5x+6} - \dfrac{2x+9}{5(x-3)} - \dfrac{3x+4}{5(x-2)}$.

43. $\dfrac{b}{(1-ab)(a-b)} + \dfrac{a}{(b^2-1)(a-b)} + \dfrac{ab^3}{(b^2-1)(1-ab)}$.

THE FRACTIONAL FORMS, $\frac{0}{a}$, $\frac{a}{0}$, $\frac{0}{0}$.

408. I. *Zero divided by a finite quantity equals zero.*

The value of any fraction will become less, if its numerator is diminished while its denominator remains constant. Hence, when the numerator is assumed less than any assignable quantity, the value of the fraction must be less than any assignable quantity. In other words, when the numerator equals zero, the value of the fraction equals zero.

Hence, $$\frac{0}{a} = 0.$$

II. *A finite quantity divided by zero equals infinity.*

The value of any fraction will become greater, if its denominator is decreased while its numerator remains constant.

Hence, when the denominator is assumed less than any assignable quantity, the value of the fraction must be greater than any assignable quantity. In other words, when the denominator equals zero, the value of the fraction equals infinity.

Hence, $$\frac{a}{0} = \infty.$$

III. *Zero divided by zero may represent any value whatever.*

Whatever the value of a,

$$a \times 0 = 0.$$

Dividing both members by 0,

$$\frac{0}{0} = a.$$

Since a may have any value whatever,

$\frac{0}{0}$ is called the **symbol of indetermination.**

SIMPLE EQUATIONS.

409. ***Find the values of x in the following:***

1. $\dfrac{8x+5}{14} + \dfrac{7x-3}{6x+2} = \dfrac{4x+6}{7}.$

2. $\dfrac{ax}{b(x+c)} + \dfrac{bx}{a(x+c)} = 1.$

3. $\frac{1}{4}[3x - \frac{2}{3}(1+x)] + \dfrac{1 - \frac{1}{6}x}{5\frac{1}{2}} = \dfrac{2\frac{2}{5} + \frac{1}{25}(x-1)}{2\frac{1}{5}}.$

4. $\dfrac{6x-7\frac{1}{2}}{13-2x}+2x+\dfrac{16x+1}{24}=4\frac{5}{12}-\dfrac{12\frac{5}{8}-8x}{3}$.

5. $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{a-b}{x-c}$.

6. $\dfrac{3+17x}{3x}+\dfrac{18+x}{5x}=\dfrac{21-x}{9x}+\dfrac{27}{4x}$.

7. $\dfrac{10x+3}{5x-4}+1=\dfrac{418}{25x^2-16}+\dfrac{15x-7}{5x+4}$.

8. $\dfrac{m^2-1}{m-n}-\dfrac{m(a-2n)}{a-x}=\dfrac{1-n^2}{n-m}-\dfrac{n(a+m)}{x-a}$.

9. $\dfrac{2}{x}-\dfrac{1}{8}+\dfrac{27x}{16}=\dfrac{17+30\frac{7}{16}x}{8x}-\dfrac{2+14\frac{7}{8}x}{16x}$.

10. $\dfrac{20+\frac{1}{3}x}{x-15}+\dfrac{19x-21}{2x+1}-9\frac{5}{6}=\dfrac{17\frac{1}{2}}{x-15}$.

11. $\dfrac{1}{x-a}-\dfrac{1}{x-b}=\dfrac{a-b}{x^2-ab}$.

12. $\dfrac{1}{x-a}-\dfrac{1}{x-a+c}=\dfrac{1}{x-b-c}-\dfrac{1}{x-b}$.

13. $\dfrac{x-a}{x-a-1}-\dfrac{x-a-1}{x-a-2}=\dfrac{x-b}{x-b-1}-\dfrac{x-b-1}{x-b-2}$

14. $(x+a)(2x+b+c)^2=(x+b)(2x+a+c)^2$.

15. $\dfrac{7-2x}{10}-\dfrac{2x-1}{5}+\frac{1}{2}x=\dfrac{5x-4\frac{2}{15}}{2x}-\dfrac{17+3x}{30}$.

16. $(x-a)^3(x+a-2b)=(x-b)^3(x-2a+b)$.

17. $\dfrac{3abc}{a+b}+\dfrac{a^2b^2}{(a+b)^3}+\dfrac{(2a+b)b^2x}{a(a+b)^2}=3cx+\dfrac{bx}{a}$.

18. $\frac{1}{3}\left(\frac{1}{6}x - 2\right) - \frac{(x-3)\left(\frac{1}{4}x - 10\right)}{9}$

$$= \tfrac{1}{6}\left(\tfrac{1}{6}x - 2\right)(42 - x).$$

19. $\dfrac{a^2(a+c)}{(a-b)(x-a)} - \dfrac{a^2(b+c)}{(a-b)(x-b)} = \dfrac{(a^3 + a^2x)\,n}{x^2 - a^2}.$

20. $5x + 1 - \left[13 + 6\left\{2x - \dfrac{3(x+3)}{23} + \dfrac{1}{5}\left(\dfrac{x+10}{6}\right)\right\}\right]$

$$= \left(6 - \frac{x}{2}\right)\left(\frac{x}{2} + 5\right) + \left(\frac{x}{4} - 9\right)x.$$

21. $\dfrac{5\left(\frac{1}{4} + x\right)}{1 + 3x} + \dfrac{17 - x}{2x + 1} = 18\tfrac{1}{4} - \dfrac{10x}{1 + 3x}.$

22. $\dfrac{3(m+n)^2}{cm^2n^2} + (m+n) = \left(\dfrac{3(m+n)}{mn}\right)\left(\dfrac{m+n}{cmn}\right)$

$$- mn + (x+1)(m+n).$$

23. $(x-a)(x-b)(x+2a+2b)$

$$= (x+2a)(x+2b)(x-a-b).$$

PROBLEMS LEADING TO EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

410. I. A cask contains 12 gallons of wine and 18 gallons of water; a second cask contains 9 gallons of wine and 3 gallons of water. How many gallons from each cask will produce a mixture composed of 7 gallons of wine and 7 gallons of water?

Let x = gallons from first cask,

then $14 - x$ = " " second cask.

Since the first cask contains 30 gallons, of which 12 gallons

are wine, a gallon from the first contains $\frac{4}{5}\frac{2}{0}$, or $\frac{2}{5}$ of a gallon of wine.

In like manner, a gallon of the second contains $\frac{9}{1}\frac{3}{2}$, or $\frac{3}{4}$ of a gallon of wine.

So that x gallons from the first contain x times $\frac{2}{5}$, or $\frac{2x}{5}$ gallons of wine; and $14 - x$ gallons from the second contain $14 - x$ times $\frac{3}{4}$, or $\frac{3\,(14 - x)}{4}$ gallons of wine.

Whence, $$\frac{2x}{5} + \frac{3\,(14 - x)}{4} = 7;$$

$$8x + 210 - 15x = 140.$$

$$-7x = -70.$$

$x = 10$, number of gallons from first cask;

$14 - x = 4$, " " " second cask.

1. Two hundred pounds of bronze contains 140 pounds of copper and 60 pounds of tin. How much pure copper must be melted with the bronze so that 84 pounds in each hundred of the mixture shall be copper?

2. Gun metal is composed of 90 per cent. of copper and 10 per cent. of tin. Speculum metal contains 67 per cent. of copper and 33 per cent. of tin. How many cwt. of speculum metal should be melted with 8 cwt. of gun metal to form an alloy containing three times as much copper as tin?

3. There are two bars of metal; the first contains 14 oz. of silver and 6 oz. of tin, the second 8 oz. of silver and 12 of tin. How much of each must be taken to form a bar of 20 oz. containing equal weights of silver and tin?

II. At what time between 5 and 6 o'clock will the hour and minute hands be together?

These time problems are simply questions of pursuit. The minute-hand goes 12 times as fast as the hour-hand. Hence, the minute-hand in *going* 12 spaces of any kind *gains* 11 of those spaces.

At 5 o'clock the minute-hand is 25 minute-spaces behind the hour-hand; it must, therefore, gain 25 minute-spaces before the hands are together.

Let x = the minute-spaces the minute-hand must go after 5 o'clock to overtake the hour-hand.

Since distances *gone* and distances *gained* by the minute-hand must always be proportional,

$$12 \text{ gone} : x \text{ gone} :: 11 \text{ gained} : 25 \text{ gained}.$$

Whence, $$11x = 300,$$

and $$x = 27\tfrac{3}{11}.$$

Hence, they are together at $27\frac{3}{11}$ minutes after 5 o'clock.

III. At what time after 8 o'clock will the minute-hand be 10 minute-spaces ahead of the hour-hand?

At 8 o'clock the minute-hand is 40 minute-spaces behind the hour-hand. Hence, it must *gain* 50 minute-spaces to be in the required position.

Let x = the minute-spaces the minute-hand must go after 8 o'clock.

Then, 12 gone : x gone :: 11 gained : 50 gained.

Whence, $11x = 600,$

and $x = 55\frac{5}{11}.$

Hence, they are in the required position at $55\frac{5}{11}$ minutes after 8 o'clock.

IV. At what time after 6 o'clock is the hour-hand just as far beyond 6 as the minute-hand lacks of being at 6 ?

Let x = the minute-spaces the hour-hand must go,

and $12x =$ " " " minute-hand "

But since in the required position the minute-hand lacks x minute-spaces of being at 6 o'clock,

$$12x + x = 30.$$

Whence, $13x = 30,$

$$x = 2\frac{4}{13},$$

and $12x = 27\frac{9}{13}.$

Hence, the required time is $27\frac{9}{13}$ minutes after 6 o'clock.

4. At what time after 2 o'clock are the hands of a clock together ?

5. At what time after 6 o'clock is the hour-hand 5 minute-spaces ahead of the minute-hand ? 10 minute-spaces behind the minute-hand ?

6. At what time between 9 and 10 o'clock are the hour and minute hands opposite ?

7. At what time between 7 and 8 o'clock is the minute-hand 15 minute-spaces ahead of the hour-hand ?

8. At what time between 11 and 12 o'clock are the hour and minute hands opposite ?

9. At what time between 7 and 8 o'clock is the hour-hand as far beyond 7 as the minute-hand lacks of being at 7 ?

10. At what time between 5 and 6 o'clock is the minute-hand as far beyond 6 as the hour-hand lacks of being at 6 ?

11. At what time between 8 and 9 o'clock is the hour-hand as far beyond 6 as the minute-hand lacks of being at 6 ?

12. Two thirds of A's money is equal to three fourths of B's, and A has $175 more than B. How much money has each ?

13. A horse and carriage cost $750, and the carriage cost $1\frac{1}{2}$ times as much as the horse. What did each cost ?

14. I paid $136 for 100 bushels of corn and 80 bushels of potatoes. How much a bushel did each cost if the corn was 10 cents a bushel more than the potatoes ?

15. A walked 24 miles farther in 12 hours than B in 9 hours. What is the rate per hour of each, if A walks as far in 4 hours as B in 5 hours ?

16. A thief has $1\frac{1}{2}$ hours' start of an officer, and travels 6 miles per hour. If the officer catches him after traveling 100 miles, what is his rate per hour ?

17. A steamboat requires 6 hours and 40 minutes longer to go up a river 100 miles than down. Find the rate of the current, if it be one-fourth that of the boat in still water.

18. If 100 gallons of a mixture of wine and water contain 8 gallons of wine, how much water must be added to the mixture so that 65 gallons may contain 4 gallons of wine ?

19. A boat steams down a river a distance of 39 miles in 2 hours and 36 minutes, and returns to the point of starting in 4 hours and 20 minutes. Find the rate of the current, and that of the steamer in still water.

20. At what time between 3 and 4 o'clock is the hour-hand as much beyond 12 as the minute-hand lacks of being at 12?

21. A thief is 20 of his own steps ahead of an officer, and takes 3 steps while the officer takes 2; but 8 of his steps are equal to 5 of the officer's steps. How many steps will each take until the thief is caught?

22. A trader maintained himself for three years, at the expense of $500 a year, and in each of those years augmented that part of his stock which was not so expended by $\frac{1}{3}$ thereof. At the end of the third year his original stock was doubled. What was his stock, at first?

23. A watch gains 2 minutes per hour more than a clock loses, and 819 hours by the watch are equivalent to 767 by the clock. Find the error of each per hour.

24. An alloy of gold and silver weighing 140 pounds contains 30% of gold; with how much silver must the mass be melted so that there may be 25% of gold in it?

25. A man leaving home took with him $\frac{4}{7}$ of his ready money; he lost $10 less than $\frac{3}{8}$ of what he had with him; paid for car-fare $13 less than $\frac{3}{10}$ of the remainder; and his pockets were picked of $14 more than $\frac{2}{5}$ of what then remained. On his return he found that he brought back only $\frac{1}{3}$ as much as he had left at home. How much money had he before leaving home?

GENERAL PROBLEMS.

411. A **general problem** is a problem in which quantities to be treated as known are represented by letters. Strictly, in a general problem, *all* values that are assumed as known are represented by letters.

412. An answer to a general problem is called a **formula.** A formula expressed in ordinary language is a **rule.**

I. The sum of two numbers is s and their difference d. What are the numbers?

Let $x =$ the greater number.

Then $x - d =$ " less "

But $x + x - d = s.$

Whence, $2x = s + d.$

And $x = \frac{s+d}{2}$, the greater number.

Also, $x - d = \frac{s+d}{2} - d$

$= \frac{s-d}{2}$, the less number.

All problems in which the sum of two numbers is given and the difference between them, may be solved by substituting the given sum and difference, respectively, for s and d in these two formulas.

Thus, if A earns \$25 more than B, and if together they earn \$71, to find what each earns,

$\frac{s+d}{2}$ becomes $\frac{\$71 + \$25}{2} = \$48$, what A earns;

and $\frac{s-d}{2}$ becomes $\frac{\$71 - \$25}{2} = \$23$, what B earns.

II. A is m times as old as B, but in b years A will be only n times as old as B. How old is each?

Let x = B's present age.

Then mx = A's " "

But $mx + b = (x + b)\,n.$

Whence, $(m - n)\,x = bn - b.$

Hence, $x = \dfrac{b\,(n-1)}{m-n}$, B's present age.

And $mx = \dfrac{bm\,(n-1)}{m-n}$, A's " "

1. A is 3 times as old as B, but in 16 years he will be only twice as old. Find by the formulas the age of each.

2. A's money is 5 times B's, but after each earns \$35, A has only 3 times as much as B. How much has each at first?

III. A can do a piece of work in a days, B in b days, and C in c days. In what time can they do it working together?

Let x = the required time.

Then $\dfrac{1}{x}$ = the part they can do in one day.

But $\dfrac{1}{a}$, $\dfrac{1}{b}$, and $\dfrac{1}{c}$ = respectively the parts A, B, and C can do in one day.

Hence, $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{x}.$

Whence, $abx + acx + bcx = abc.$

And $$x = \frac{abc}{ab + ac + bc}.$$

3. A, B, and C respectively can do a piece of work in 8, 12, and 15 days. In what time can they do it together?

4. A cistern has three pipes. Separately, the first can fill it in 4 hours, the second in 6 hours, and the third in 12 hours. How long will it take if they run together?

IV. A steamer whose rate in still water is a miles per hour, can go a certain distance down stream in m hours and return in n hours. What is the velocity of the current?

Let $x =$ velocity of the current per hour.

Then $a + x =$ rate per hour down stream.

And $a - x =$ " " " up "

So that $(a + x)\, m = (a - x)\, n.$

Whence, $$x = \frac{a\,(n - m)}{n + m}.$$

5. A steamer whose rate in still water is 15 miles per hour can go a certain distance down stream in 6 hours and return in 9 hours. What is the velocity of the current?

To show more fully the nature of generalization, we shall solve a celebrated problem and discuss the result.

PROBLEM OF THE COURIERS.

V. Two couriers, P and Q, travel at uniform rates along a road AB in the same direction. P's rate is a miles per hour,

and Q's b miles per hour. At a certain time they are m miles apart; when are they together?

Let x = the No. miles Q travels before he is overtaken.

Then $m + x$ = the No. miles P travels to overtake Q.

Hence, $\frac{x}{b}$ = time Q travels before he is overtaken.

And $\frac{m+x}{a}$ = " P " to overtake Q.

So that, $\frac{x}{b} = \frac{m+x}{a}$, since the times are equal.

Whence, $ax = bm + bx$.

And $x = \frac{bm}{a-b}$, distance Q travels.

Also, $m + x = m + \frac{bm}{a-b}$

$= \frac{am}{a-b}$, distance P travels.

The time, $\frac{x}{b} = \frac{bm}{a-b} \div b = \frac{m}{a-b}$.

In discussing these formulas, the distance between the couriers and their rates of travel may be assumed at pleasure, that is, they are *arbitrary* quantities.

1. a may be *greater* than b,
2. a " " *less* " b,
3. a " " *equal to* b,

} when m is greater than 0.

4. a may be *greater* than b,
5. a " " *less* " b,
6. a " " *equal to* b,

} when m is equal to 0.

In this discussion, when the formula for the time becomes negative for any values of a, b, and m, it will be interpreted to mean *time before* the "certain time" mentioned in the problem (T, for brevity); when positive, it will be interpreted to mean *time after* T (Art. **52**).

In like manner, *positive values* for the distance formula will mean *distance to the right* of Q's position at T; *negative values* will mean *distance to the left* of Q's position at T.

In (1),

$\frac{m}{a-b}$ is *positive;* they will be together *after* T.

$\frac{bm}{a-b}$ " " " " at the *right.*

Since they are together ***after*** *T, it must be to the right of Q's position at T.*

In (2),

$\frac{m}{a-b}$ is *negative;* they were together before T.

$\frac{bm}{a-b}$ " " " " at the left.

Since they were together ***before*** *T, it must have been to the left of Q's position at T.*

In (3),

$\frac{m}{a-b}$ becomes ∞ ; the *time* is infinite.

$\frac{bm}{a-b}$ " ∞ ; the *distance* is infinite.

Since their rates are equal, and they are separated at T, they never have been, and never will be, together at any point of time or distance.

In (4),

$\frac{m}{a-b}$ becomes $+0$, the *time* after T when they are together.

$\frac{bm}{a-b}$ " $+0$, the *distance* to the right when they are together.

Since P's *rate exceeds* Q's, P *passes* Q *at T; they have never been together before, and will never be together again.*

In (5),

$\frac{m}{a-b}$ becomes -0, the *time before T* when they are together.

$\frac{bm}{a-b}$ " -0, the *distance to the left.*

Since Q's *rate exceeds* P's, Q *passes* P *at T; they have never been together before, and will never be together again.*

In (6),

$\frac{m}{a-b}$ becomes $\frac{0}{0}$; the time is indeterminate.

$\frac{bm}{a-b}$ " $\frac{0}{0}$; the distance is indeterminate.

Since P's *rate equals* Q's, *and they are together at T, they always have been, and always will be, together.*

6. Find, by substitution in the foregoing formulas, at what time between 5 and 6 o'clock the hands of a clock will be together.

7. A train whose rate is 30 miles an hour leaves New York, and is followed 4 hours later by a train whose rate is 36 miles an hour. How far from New York, and how long after the faster train starts, will it overtake the slower train?

413. ***Solve the following general problems:***

1. Divide d dollars among four brothers so that each shall have m dollars more than the next younger.

2. One man spent a dollars for sheep, and another got the same number of sheep for b dollars, paying c dollars less per head. What price did each pay?

3. Divide the number n into four such parts that the first increased by r, the second diminished by r, the third multiplied by r, and the fourth divided by r shall all be equal.

4. A has a saddle worth a dollars, and two horses. If the saddle be put on the first horse, their united value will be m times that of the second horse; if it be put on the second horse, their united value will be n times the value of the first horse. What is the value of each horse?

5. A person rides in a coach at the rate of m miles an hour and walks back at the rate of n miles per hour. How far did he go if he was gone h hours?

6. The length of a field is a times its breadth; another field b rods longer and c rods wider contains d acres more than the former. What are the dimensions of the smaller field?

7. Two vessels contain equal quantities of water. From the first a gallons are drawn, and from the second b gallons, after which m times as much remains in the second as in the first. How much did each originally contain?

8. How much tea worth a cents a pound must be mixed with b pounds worth c cents a pound so that the mixture will be worth d cents a pound?

9. A workman is engaged for a days at b dollars a day, on condition that he shall forfeit c dollars for each day he is idle. At the end of the time he receives d dollars. How many days does he work?

10. A garrison of m men had provisions for d days. At the end of e days a reinforcement came, and the provisions lasted only r days longer. How many men in the reinforcement?

11. A, B, C, and D invest a dollars in a speculation. B furnishes m times as much as A; C as much as A and B together; D as much as B and C together. How much does each invest?

12. A and B can do a piece of work in a days; A and C in b days; B and C in c days. How long would it take each to do it? How long all together?

13. Gold is a times as heavy as water, and silver is b times as heavy. A mixed mass weighs c ounces and displaces d ounces of water. How many ounces of gold in the mixture?

MISCELLANEOUS RADICAL EQUATIONS.

414. ***Find the value of x:***

1. $\dfrac{5x-9}{\sqrt{5x}+3}-1=\dfrac{\sqrt{5x}-3}{2}.$

2. $\dfrac{3x-1}{\sqrt{3x}+1}=1+\dfrac{\sqrt{3x}-1}{2}.$

3. $\sqrt{x-4}=\dfrac{8-x}{\sqrt{x+4}}.$

4. $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1.$

5. $\sqrt{2ax+x^2}=a-x.$

6. $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}.$

7. $x+a+\sqrt{2ax+x^2}=b.$

8. $\sqrt{a+x}+\sqrt{a-x}=2\sqrt{x}.$

9. $\sqrt{5\,(x+2)}=\sqrt{5x}+2.$

10. $\sqrt{x}-\sqrt{a+x}=\dfrac{\sqrt{a}}{\sqrt{x}}.$

11. $\sqrt{bx}+\sqrt{b\,(a+x)}=\sqrt{x}.$

12. $\sqrt{bx+x^2}=1+x.$

13. $\sqrt{x-a}=\sqrt{x+\sqrt{b+x}}.$

14. $a+x-\sqrt{a^2+x^2}=b.$

15. $\sqrt{x + 2\sqrt{ax + a^2}} = \sqrt{a} + \sqrt{x}$.

16. $a + x - \sqrt{2ax + x^2} = b$.

17. $a + x + \sqrt{a^2 + bx + x^2} = b$.

18. $\sqrt[4]{b^4 + x^4} = \sqrt{a^2 + x^2}$.

19. $\dfrac{1}{a - \sqrt{a^2 - x^2}} - \dfrac{1}{a + \sqrt{a^2 - x^2}} = \dfrac{a}{x^2}$.

20. $\dfrac{1}{x + \sqrt{2 - x^2}} + \dfrac{1}{x - \sqrt{2 - x^2}} = ax$.

21. $x + \sqrt{a^2 + x^2} = \dfrac{na^2}{\sqrt{a^2 + x^2}}$.

22. $\sqrt{1 + x^2 + \sqrt{1 - x^2}} = \sqrt{1 - x^2} - \sqrt{1 + x^2}$.

23. $\dfrac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} = c$.

24. $\dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{x}} = \dfrac{\sqrt{x}}{\sqrt{b}}$.

25. $\dfrac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} + \dfrac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = 34$.

26. $\dfrac{x + \sqrt{x^2 - a^2}}{x - \sqrt{x^2 - a^2}} = \dfrac{x}{a}$.

27. $2\sqrt{2a^2 + ax + x^2} = x + 3a$.

28. $\sqrt{a + x} + \sqrt{a - x} = \dfrac{12a}{5\sqrt{a + x}}$.

29. $\dfrac{\sqrt{x}}{\sqrt{a}} + \dfrac{\sqrt{a}}{\sqrt{x}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.

30. $\dfrac{x}{\sqrt{x} + \sqrt{a-x}} + \dfrac{x}{\sqrt{x} - \sqrt{a-x}} = \dfrac{b}{\sqrt{x}}$.

31. $\dfrac{a + x + \sqrt{2ax + x^2}}{a + x - \sqrt{2ax + x^2}} = b^2$.

32. $\dfrac{\sqrt{\dfrac{a+x}{a-x}} + b}{\sqrt{\dfrac{a+x}{a-x}} - b} = \dfrac{3}{2}$.

33. $\dfrac{\sqrt{3x+1} + 3}{\sqrt{3x+1} - 3} = \dfrac{\sqrt{7x+8} + 4\frac{1}{2}}{\sqrt{7x+8} - 4\frac{1}{2}}$.

34. $\dfrac{\sqrt{3x+1} + \sqrt{2x-7}}{\sqrt{3x+1} - \sqrt{2x-7}} = \dfrac{\sqrt{28x+1} + \sqrt{3(4x-5)}}{\sqrt{28x+1} - \sqrt{3(4x-5)}}$.

35. $\dfrac{\sqrt{6x+1} + \sqrt{2x}}{\sqrt{6x+1} - \sqrt{2x}} = \dfrac{7\sqrt{x-4} + \sqrt{7x+8}}{7\sqrt{x-4} - \sqrt{7x+8}}$.

36. $\dfrac{\sqrt{3(x+1)} + \sqrt{x-2}}{\sqrt{3(x+1)} - \sqrt{x-2}} = \dfrac{\sqrt{9x+1} + \sqrt{2x+3}}{\sqrt{9x+1} - \sqrt{2x+3}}$.

37. $\dfrac{\sqrt{4x+5} + \sqrt{2x+3}}{\sqrt{4x+5} - \sqrt{2x+3}} = \dfrac{\sqrt{5x-6} + \sqrt{3x-8}}{\sqrt{5x-6} - \sqrt{3x-8}}$.

38. $\dfrac{\sqrt{a+x} + \sqrt{b-x}}{\sqrt{a+x} - \sqrt{b-x}} = \dfrac{\sqrt{c+x} + \sqrt{a-x}}{\sqrt{c+x} - \sqrt{a-x}}$.

MISCELLANEOUS QUADRATIC EQUATIONS.

415. ***Solve the following:***

1. $\frac{2+x^2}{3}+\frac{x^2-x}{2}=1-x+x^2.$

2. $\frac{2x}{15}+\frac{3x-50}{3(x+10)}=\frac{6x+35}{95}.$

3. $6x+\frac{35-3x}{x}=44.$

4. $\frac{x}{7}+\frac{21}{x+5}=6\frac{4}{7}.$

5. $\frac{2}{x-1}+\frac{12}{x^2-1}=1.$

6. $\frac{2x+11}{x}+\frac{x-5}{3}=5.$

7. $\frac{2x}{x+2}+\frac{x+2}{2x}=2.$

8. $\frac{x}{x+1}+\frac{x}{x+4}=1.$

9. $\frac{x+1}{x-1}-\frac{x-2}{x+2}=1\frac{1}{8}.$

10. $\frac{x+4}{x-4}-\frac{x+2}{2-x}=7.$

11. $\frac{x-1}{x-4}-\frac{x-3}{x-2}=\frac{11}{12}.$

12. $\frac{1}{x-2}-\frac{2}{x+2}=\frac{3}{5}.$

13. $\frac{x-2}{x-3}-\frac{x-4}{x-1}=\frac{14}{15}.$

14. $\frac{x-3}{x-2}+1\frac{1}{3}=\frac{x-1}{x-4}.$

15. $\frac{x+2}{x-2}+\frac{x-2}{x+2}=\frac{5x+33}{x^2-4}.$

16. $\frac{3(x-1)}{x+1}-\frac{2(x+1)}{x-1}-5=0.$

17. $\frac{2x+1}{x-1}-\frac{2-3x}{2+3x}=5\frac{1}{2}.$

18. $\frac{3x-2}{2x-5}-\frac{5-2x}{3x-2}=3\frac{1}{3}.$

19. $\frac{x+4}{3} - \frac{7-x}{x-3} - \frac{4x+7}{9} = -1.$

20. $\frac{x^2-4}{7} + \frac{9x+3}{16} - \frac{9x-3}{21} = \frac{(x-1)^2}{4}.$

21. $\frac{3x-5}{9x} - \frac{6x}{3x-25} = \frac{1}{3}.$

22. $\frac{3x+4}{5} - \frac{30-2x}{x-6} = \frac{7x-14}{10}.$

23. $\frac{(x+7)(x-4)}{(x+4)(x-7)} - \frac{40}{13} = 0.$

24. $13 - \frac{2x+3}{5x} - \frac{x-2}{x+1} = 11.$

25. $\frac{x(x-1)}{(x+1)(x+2)} = \frac{(x+3)(x+4)}{(x+5)(x+6)}.$

26. $\frac{x+7}{x+11} - \frac{x+5}{x+12} = \frac{47}{306}.$

27. $\frac{x}{x+5} - \frac{11x}{11x-8} + \frac{7}{6-4x} = 0.$

28. $8(xy+1) = 33y;\quad 4(xy+1) = 33x.$

29. $x^3 - y^3 = 7930;\quad x - y = 10.$

30. $x^2(x-y) = 75;\quad x^2(2x+3y) = 400.$

31. $x^2 - xy + y^2 = 49 - 2xy;\quad x^4 + x^2y^2 + y^4 = 931.$

32. $x^2 + y^2 + 4(x^2+y^2)^{\frac{1}{2}} = 45;\quad x^4 + y^4 = 337.$

33. $x^2 + 2xy + y^2 + 2x = 120 - 2y;\quad xy - y^2 = 8.$

34. $3x^2 + xy = 68;\quad 4y^2 + 3xy = 160.$

35. $x^2y^4 + y^2 = 333;\quad xy^2 + y = 21.$

36. $x^4 + x^2y^2 + y^4 = 133;\quad x^2 - xy + y^2 = 7.$

INEQUALITIES.

416. An **inequality** is a statement, in algebraic language, that one quantity is greater or less than another quantity.

417. The fact of inequality is expressed by the signs $>$ or $<$, the opening of the sign being toward the greater quantity.

Thus, $a > b$, read, *a is greater than b*; $a < b$, read, *a is less than b*.

418. One quantity, a, is greater or less than another quantity, b, according as $a - b$ is positive or negative.

Thus, the values of a and b being finite and unequal, $a^2 + b^2 > 2ab$, since $a^2 + b^2 - 2ab = (a - b)^2$, which is positive. (Art. **239**.)

When $a = b$, the inequality becomes an equation.

Also, $1 + 4x < x^2 + 6$, since $1 + 4x - (x^2 + 6) = 4x - x^2 - 5 = -(x - 2)^2 - 1$, a negative quantity.

PROPERTIES OF INEQUALITIES.

419. I. *An inequality will not be reversed by the addition or subtraction of the same quantity from each member.*

For, if $a > b$, $a - b$ is positive. Hence, $a \pm c - (b \pm c)$ is also positive. That is, $a \pm c > b \pm c$.

From the foregoing it follows that,

A term may be transposed from one member of an inequality to the other, provided its sign is changed.

Thus, if $a + b > c - d$, by adding $+ d$ to each member, $a + b + d > c$.

II. *An inequality will not be reversed when both members are multiplied or divided by the same positive quantity.*

For, if $a > b$, $a - b$ is positive. Let m be a positive multiplier; then $m(a - b)$, or $am - bm$, is positive. That is, $am > bm$.

Also, $\frac{a}{m} - \frac{b}{m}$ is positive. That is, $\frac{a}{m} > \frac{b}{m}$.

III. *An inequality consisting of positive quantities will not be reversed by raising both members to the same power.*

For, if $a > b$, multiplying both members by a gives $a^2 > ab$, and multiplying both members by b gives $ab > b^2$.

Since $a^2 > ab$, and $ab > b^2$, it follows that $a^2 > b^2$.

IV. *An inequality will be reversed by multiplying or dividing both members by the same negative quantity.*

For, if $a > b$, $am - bm$ will be positive and $bm - am$ negative. That is, $-am < -bm$.

Also, if $-a > -b$, $-a-(-b)$ or $-a+b$ is positive, and $a - b$ is negative, as is also $am - bm$. But multiplying $-a$ and $-b$ by $-m$ gives am and bm.

Hence, $am < bm$. (The proof in the case of division is similar.)

It follows that,

An inequality will be reversed by changing the signs of all its terms.

V. *The sum of the greater members of several inequalities will exceed the sum of the lesser members.*

Thus, if $a > b$, $c > d$, $e > f$, etc., it is obvious that $(a + c + e + \text{etc.}) > (b + d + f + \text{etc.})$.

VI. *If the members of one inequality be subtracted from the non-corresponding members of another, the sign of the result will be in the same direction as the sign of the minuend.*

Thus, if $a > b$ and $c > d$, then $(a - d) > (b - c)$.

Also, if $m < n$ and $p > q$, then $(m - p) < (n - q)$.

For, since $a > b$ and $c > d$, $a - b$ and $c - d$ are positive.

Hence, their sum, $a - b + c - d$, or $(a - d) - (b - c)$ is positive.

That is, $(a - d) > (b - c)$.

Again, since $m < n$ and $p > q$, $m - n$ and $q - p$ are negative.

Hence, their sum, $m - n + q - p$, or $(m - p) - (n - q)$ is negative.

That is, $(m - p) < (n - q)$.

EXERCISES.

420. I. Required to find the limit of x in

$$5x - 4\tfrac{4}{5} > 33\tfrac{1}{3} - 2\tfrac{1}{15}x.$$

Transposing, $7\tfrac{1}{15}x > 38.$

Clearing of fractions, $38x > 190.$

Whence, $x > 5.$

II. Required to prove that $a^2 + b^2$ is equal to or greater than $2ab$, according as a and b are equal or unequal.

When a is greater or less than b, $a^2 - 2ab + b^2$ is positive.

That is, $a^2 - 2ab + b^2 > 0.$

Transposing, $a^2 + b^2 > 2ab.$

When $a = b$, the expression $a^2 - 2ab + b^2$ reduces to zero.

III. Required to divide the number $2a$ into two parts whose product shall be a maximum.

Let x be one part; then $2a - x$ will be the other part.

Calling their product y,

$$(2a - x)\,x = y.$$

Whence, $x^2 - 2ax = -y.$

And $x = a \pm \sqrt{a^2 - y}.$

That x may be real, y must not be greater than a^2.

Hence, $x = a$ when the product y is the greatest possible; that is, equal to a^2.

1. Given $7x - 7\frac{1}{3} > \frac{2x}{3} + 5$; to find the limit of x.

2. Given $x + \frac{x}{2} + \frac{x}{3} > 11$; to find the limit of x.

3. Given $\frac{x}{2} + 3x - 5 > 16$; to find the limit of x.

4. Given $7x - 1 > 34$; to find the limit of x.

5. Given $\frac{a^2 - b^2}{6d} > \frac{a^2 - b^2}{3x}$; to find the limit of x.

6. Find the positive value of x that will render $x + \frac{1}{x}$ the least possible.

7. Prove that $x^2 - 8x + 20 > 4$.

8. Given $\frac{ax}{5} + bx - ab > \frac{a^2}{5}$; to find the limit of x.

9. Given $\frac{bx}{7} - ax + ab < \frac{b^2}{7}$; to find the limit of x.

10. Given $\left\{\begin{array}{l} 3x - 4 < x + 6 \\ 5x + 7 > 3x + 13 \end{array}\right\}$; to find an integral value of x.

11. Given $\left\{\begin{array}{l} \frac{1}{4}(x + 2) + \frac{1}{3}x < \frac{1}{2}(x - 4) + 3 \\ \frac{1}{4}(x + 2) + \frac{1}{3}x > \frac{1}{2}(x + 1) + \frac{1}{3} \end{array}\right\}$; to find an integral value of x.

12. Three times a certain whole number, increased by 2, is more than twice the number, increased by 61; and 5 times the number, diminished by 70, is less than 4 times the number, diminished by 9. What is the number?

13. The sum of two numbers is 25; if the greater be divided by the less, the quotient will exceed 3; and if the less be divided by the greater, the quotient will exceed $\frac{1}{3}$. What are the numbers?

PROPERTIES OF QUADRATIC EQUATIONS.

421. *A quadratic equation cannot have more than two roots.*

Every quadratic equation may be reduced to the form

$$x^2 + px + q = 0,$$

in which p and q may be positive or negative, fractional or integral.

Let it be supposed that there can be *three* roots, m, n, and r, unequal in value.

Then will
$$m^2 + pm + q = 0; \tag{1}$$
$$n^2 + pn + q = 0; \tag{2}$$
$$r^2 + pr + q = 0. \tag{3}$$

Subtracting (2) from (1), and (3) from (1),

$$(m^2 - n^2) + (m - n)p = 0. \tag{4}$$
$$(m^2 - r^2) + (m - r)p = 0. \tag{5}$$

Dividing (4) by $m - n$, and (5) by $m - r$,

$$(m + n) + p = 0. \tag{6}$$
$$(m + r) + p = 0. \tag{7}$$

Subtracting (7) from (6),

$$n - r = 0.$$

Whence,
$$n = r.$$

But, by supposition, n and r are unequal.
Hence, there cannot be more than two roots.

422. *The sum of the two roots of an equation in the form* $x^2 + px + q = 0$ *is equal to the co-efficient of the second term with its sign changed, and their product is equal to the third term.*

The two roots are

$$-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q} \quad \text{and} \quad -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}.$$

Adding these roots gives $-p$.

Multiplying them together gives

$$\frac{p^2}{4} - \left(\frac{p^2}{4} - q\right) = q.$$

423. *The general equation* $x^2 + px + q = 0$ *may be separated into two binomial factors* $(x - r)(x - r')$, *in which* r *and* r' *are its two roots.*

For, since r and r' are the roots,

$$r + r' = -p, \quad \text{or} \quad -(r + r') = p,$$

and

$$rr' = q.$$

Substituting these in the general equation,

$$x^2 - (r + r')x + rr' = 0.$$

But this, by Art. **130**, equals

$$(x - r)(x - r') = 0.$$

I. Required to write the quadratic equation whose roots added together equal -2, and multiplied together equal -15.

Since the sum, -2, is, in the general equation, the coefficient p taken with the contrary sign, and the product, -15, is the third term, q, the equation is

$$x^2 + 2x - 15 = 0.$$

II. Required to find the equation whose roots are 7 and -5.

From the principle explained in Art. **423**, the equation, whatever it is, is separable into two factors,

$$\{x - (+7)\}\{x - (-5)\} = (x - 7)(x + 5).$$

Hence, the equation is

$$x^2 - 2x - 35 = 0.$$

III. Required to find by factoring the roots of the equation,

$$x^2 - 8x + 15 = 0.$$

Factoring as in Art. **130**,

$$x^2 - 8x + 15 = (x - 5)(x - 3).$$

Hence, $$(x - 5)(x - 3) = 0.$$

This equation is satisfied by making either factor equal to 0.

When $x - 5 = 0$, $x = 5$.

When $x - 3 = 0$, $x = 3$.

424. This method is applicable to equations of higher degree than the second whenever the left member is separable into factors. Thus,

IV. Required to solve the equation $x^3 - 9x^2 + 23x - 15 = 0$. Also, $6x^2 + x - 77 = 0$.

Factoring as in Art. **136**,

$$x^3 - 9x^2 + 23x - 15 = (x - 1)(x - 5)(x - 3) = 0.$$

Whence, $x = 1, 5,$ and 3.

Factoring as in Art. **132**,

$$6x^2 + x - 77 = (2x - 7)(3x + 11) = 0.$$

Whence, $2x = 7$, and $x = 3\frac{1}{2}$.

Again, $3x = -11$, and $x = -3\frac{2}{3}$.

V. Required to solve the equation $x^3 - 4x^2 - 2x + 15 = 0$.

Factoring as in Art. **134**,

$$x^3 - 4x^2 - 2x + 15 = x^2(x - 3) - x(x - 3) - 5(x - 3)$$
$$= (x^2 - x - 5)(x - 3) = 0.$$

When $x - 3 = 0$, $x = 3$.

When $x^2 - x - 5 = 0$, $x = \frac{1}{2}(1 \pm \sqrt{21})$.

425. It follows from the foregoing that any quadratic expression may be resolved into factors by making it equal to zero, and finding its roots. Thus,

VI. Required to factor $x^2 - 2ax + a^2 - b^2$.

Let $x^2 - 2ax + a^2 - b^2 = 0$.

Whence, $x = a + b$, or $a - b$.

Hence,

$$x^2 - 2ax + a^2 - b^2 = \{x - (a + b)\}\{x - (a - b)\}$$
$$= (x - a - b)(x - a + b).$$

EXERCISES.

426. *1.* Write the equation in which

$$p = 2a, \quad \text{and} \quad q = a^2 - b^2.$$

2. Write the equation in which

$$p = \frac{2(a^2 + b^2)}{b^2 - a^2}, \quad \text{and} \quad q = 1.$$

3. One of the roots of a quadratic equation is

$$a + \sqrt{a^2 - b^2},$$

and $q = b^2$. What is the other root?

4. Write the equation whose roots are 3 and 11. 4 and -1. $\frac{2}{3}$ and $\frac{3}{2}$.

5. Write the equation whose roots are $a + b$ and $a - b$.

6. Write the equation whose roots are $2 + \sqrt{3}$ and $-2(2 + \sqrt{3})$.

Find by the method of factoring the roots of the following:

7. $x^2 + 3x - 10 = 0$.

8. $x^2 + 7x + 12 = 0$.

9. $x^2 - 8x = 65$.

10. $5x^2 + 11x + 2 = 0$.

11. $2x^2 + 11x + 15 = 0$.

12. $39x^2 - 34x + 7 = 0$.

13. $21x^2 + 11x - 2 = 0$.

14. $35x^2 - 2x - 117 = 0$.

15. $22x^2 - 41x - 35 = 0$.

16. $8x^2 - 19x - 27 = 0$.

17. $x^3 + x^2 - 17x + 15 = 0.$

18. $x^3 + 10x^2 + 31x + 30 = 0.$

19. $x^3 - 10x^2 + 23x - 14 = 0.$

20. $x^3 + x^2 - 65x + 63 = 0.$

21. $x^3 + 3x^2 - 10x - 24 = 0.$

22. $x^3 + 6x^2 - 24x - 64 = 0.$

23. $x^3 - 19x - 30 = 0.$

24. $x^3 - 6x^2 - 67x - 60 = 0.$

25. $3x^3 - x^2 - 19x - 15 = 0.$

26. $x^3 - 14x^2 + 35x - 22 = 0.$

GENERAL DISCUSSION OF QUADRATIC EQUATIONS.

427. In the general equation, $x^2 + px = q$, since p and q may be either positive or negative, four forms of the equation are needed to represent all its varieties with respect to signs.

EQUATIONS.		ROOTS.	
$x^2 + px = +q.$	(1)	$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}.$	(1)
$x^2 + px = -q.$	(2)	$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$	(2)
$x^2 - px = +q.$	(3)	$x = +\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}.$	(3)
$x^2 - px = -q.$	(4)	$x = +\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$	(4)

428. Under various suppositions with respect to p and q, there may be found among these forms every possible variety of roots—*positive* or *negative, real* or *imaginary, equal* or *unequal.*

I. POSITIVE AND NEGATIVE ROOTS.

The value of $\sqrt{\frac{p^2}{4}+q}$ is greater than $\frac{p}{2}$, and the value of $\sqrt{\frac{p^2}{4}-q}$ is less than $\frac{p}{2}$, since $\sqrt{\frac{p^2}{4}}=\frac{p}{2}$. Hence,

In form
(1) *One root is positive and one negative.*
(2) *Both roots are negative.*
(3) *One root is positive and one negative.*
(4) *Both roots are positive.*

II. REAL AND IMAGINARY ROOTS.

When $\sqrt{\frac{p^2}{4}-q}$ is positive, that is, when $\frac{p^2}{4}>q$, its value is real; when negative, that is, when $\frac{p^2}{4}<q$, its value is imaginary. The value of $\sqrt{\frac{p^2}{4}+q}$ is always real. (Art. **240.**) Hence,

In forms
(1), (3) *Both roots are always real.*
(2), (4) *Both roots are real when* $\frac{p^2}{4}>q$; *otherwise both are imaginary.*

III. EQUAL AND UNEQUAL ROOTS.

When $\frac{p^2}{4}$ is equal to q, $\sqrt{\frac{p^2}{4} - q}$ reduces to zero, but $\sqrt{\frac{p^2}{4} + q}$ is still a positive quantity. Hence,

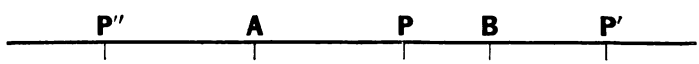
In forms $\left\{\begin{matrix}(1)\\(3)\end{matrix}\right\}$ *Both roots in each form are unequal.*

$\left\{\begin{matrix}(2)\\(4)\end{matrix}\right\}$ *Both roots in each form are equal; taking the forms* $-\frac{p}{2} \pm 0$, *and* $+\frac{p}{2} \pm 0$, *respectively.*

429. The nature and significance of the roots of quadratic equations are elegantly shown in the discussion of

THE PROBLEM OF THE LIGHTS.

Two lights, A and B, whose intensities at the distance 1 are a and b respectively, are at A and B, at a distance d from each other. It is required to find the point on the indefinite line that joins them, that is equally illuminated by them.



Assume A as the origin of distances, and P the required point of equal illumination.

Let x = the distance AP.

Then $d - x$ = the distance PB.

Since the intensity of light varies inversely as the square of the distance, and since the illumination from each light is to be the same at P,

$$\frac{a}{x^2} = \frac{b}{(d-x)^2}.$$

Whence, $$\frac{(d-x)^2}{x^2} = \frac{b}{a}.$$

And $$\frac{d-x}{x} = \pm\frac{\sqrt{b}}{\sqrt{a}}.$$

Solving this equation,

$$x = \frac{d\sqrt{a}}{\sqrt{a}+\sqrt{b}} = d\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right),$$

and $$x = \frac{d\sqrt{a}}{\sqrt{a}-\sqrt{b}} = d\left(\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\right).$$

Since these values of x are both real, there must be two points of equal illumination on the indefinite line joining the lights.

In discussing these formulas, a, b, and d may be assumed at pleasure.

1. a may be greater than b,
2. a " " less " b,
3. a " " equal to b,

} when d is greater than 0.

4. a may be greater than b.
5. a " " less " b,
6. a " " equal to b,

} when d is equal to 0.

In (1),

$d\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)$ is *positive;* hence, the point is at the *right* of the origin A.

It is greater than $\frac{d}{2}$ and less than d, since $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ is more than $\frac{1}{2}$ and less than 1. Hence, the point is *between* A and B, but *farther* from A than from B.

$d\left(\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\right)$ is *positive;* hence, the point is at the *right* of the origin A.

It is greater than d, since $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ is greater than 1. Hence, the point beyond B is at the *right* of the weaker light.

These results are as they should be; for,

Since the stronger light is at A, there must be a point between the lights, and a point at the right of the weaker light, equally illuminated.

In (2),

$d\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)$ is *positive;* hence, the point is at the *right* of the origin.

It is greater than 0, but less than $\frac{d}{2}$, since $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ is greater than 0 but less than $\frac{1}{2}$. Hence, the point is *between* A and B, but nearer to A than to B.

$d\left(\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\right)$ is *negative;* hence the point is at the *left* of the origin.

The *value* of $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ depends on the relative values of a and b; being numerically greater or less than -1, according as $\sqrt{b}$ is greater or less than $2\sqrt{a}$.

These results are consistent with the conditions; for,

Since the stronger light is at B, *there must be a point between the lights, and a point at the left of the weaker light, equally illuminated.*

In (3),

$d\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)$ is *positive;* hence, the point is at the *right* of the origin.

Its value is equal to $\frac{d}{2}$; hence, the point is *midway between* the lights.

$d\left(\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\right)$ reduces to $\frac{d\sqrt{a}}{\pm 0} = \pm\infty$; hence, the points of equal illumination are at an infinite distance to the right and the left of the origin. That is to say, this value fixes no point.

If, however, a and b are conceived as unequal and as approaching each other in value, the points of equal illumination determined in the second parts of (1) and (2) will recede to the right or to the left from A, according as a is greater or

less than b. When the difference between a and b is less than any assignable value, the points will be at a greater than any assignable distance.

In (4) and (5),

The values become

$$\frac{0}{\sqrt{a}+\sqrt{b}} = +0, \quad \text{and} \quad \frac{0}{\sqrt{a}-\sqrt{b}} = -0.$$

In these cases the lights are at the origin, A, and are unequal. The foregoing values of x seem to fix the point of equal illumination at a distance from A equal to zero. But since the lights are of unequal intensity, this cannot be.

Let the unequal lights, however, be conceived of as separated by a distance d, and as approaching each other. The first value of x, which fixes the point of equal illumination *between* the lights, and the second value of x, which fixes that beyond the weaker light, become less and less as d diminishes. When d differs from zero by less than any assignable quantity, the points of equal illumination are distant from the origin by less than any assignable distance. But when d becomes *exactly* zero, there is no equal illumination.

In (6),

The values become $\frac{0}{2\sqrt{a}} = 0$, and $\frac{0}{0}$.

The first result indicates that there is equal illumination at the *origin*; which is the fact.

The second indicates that there is equal illumination at every point along the line; which is also true.

INDETERMINATE EQUATIONS.

430. An equation containing more than one unknown quantity may be satisfied by an unlimited number of roots.

Thus, $2x + 3y = 30$ is satisfied if

$$\left\{\begin{matrix} x = 1 \\ y = 9\frac{1}{3} \end{matrix}\right\}; \quad \left\{\begin{matrix} x = 2 \\ y = 8\frac{2}{3} \end{matrix}\right\}; \quad \left\{\begin{matrix} x = 3 \\ y = 8 \end{matrix}\right\}; \quad \left\{\begin{matrix} x = 4 \\ y = 7\frac{1}{3} \end{matrix}\right\};$$

etc.

431. If it be required, however, that the roots shall be *positive integers*, it may be shown that,

I. *An equation of the general form* $ax - by = \pm c$ *is capable of an indefinite number of solutions in positive integers, if* a *and* b *are prime to each other.*

II. *An equation of the general form* $ax + by = c$ *is sometimes capable of one or more solutions in positive integers, if* a *and* b *are prime to each other and* c *is greater than* $ab - a - b$.

EXERCISES.

432. To solve in positive integers an indeterminate equation between two unknown quantities.

I. Required to solve in positive integers $5x - 8y = 7$.

Transposing and dividing by co-efficient of x,

$$x = \frac{8y + 7}{5}. \qquad (1)$$

Performing the division in second member of (1),

$$\frac{8y+7}{5} = y + 1 + \frac{3y+2}{5}.$$

Since x is integral, $\frac{3y+2}{5}$ and $\frac{2(3y+2)}{5}$ are also integral.

But $$\frac{6y+4}{5} = y + \frac{y+4}{5}.$$

Since $\frac{6y+4}{5}$ is integral, $\frac{y+4}{5}$ is also integral.

Let $\frac{y+4}{5} = m$; then

$$y = 5m - 4. \tag{2}$$

Substituting in (1) the value of y in (2),

$$x = 8m - 5. \tag{3}$$

Assuming successive integral values for m in (2) and (3) gives,

If $m = 1$, $\left\{\begin{matrix} x = 3 \\ y = 1 \end{matrix}\right\}$; if $m = 2$, $\left\{\begin{matrix} x = 11 \\ y = 6 \end{matrix}\right\}$;

if $m = 3$, $\left\{\begin{matrix} x = 19 \\ y = 11 \end{matrix}\right\}$; etc.

II. Required to solve $5x + 7y = 29$, in positive integers.

Transposing, etc., $$x = \frac{29 - 7y}{5}. \tag{1}$$

Dividing in second member,

$$x = 5 - y + \frac{2(2-y)}{5}.$$

Since x is integral, $\frac{2-y}{5}$ must be integral or equal to zero.

Let $\frac{2-y}{5} = m$; then

$$y = 2 - 5m. \tag{2}$$

Substituting in (1), $x = 3 + 7m.$ (3)

Making $m = 0$,

$x = 3$, $y = 2$, the only values possible.

III. Required to find the least number which, divided by 13, 17, and 23, leaves remainders of 7, 11, and 13, respectively.

Let $x =$ the number.

Then $\frac{x-7}{13}$ (1), $\frac{x-11}{17}$ (2), and $\frac{x-13}{23}$ (3) are integers.

Let $\frac{x-7}{13} = m$; then

$$x = 13m + 7. \tag{4}$$

Substituting in (2) the value of x in (4), gives

$$\frac{13m-4}{17}. \tag{5}$$

To avoid fractions in the value of m, it is necessary to multiply the numerator $13m - 4$ by a number such that there

will be a remainder of 1 when the co-efficient of m is divided by 17. A brief inspection will show that 4 is the multiplier required. The reasoning proceeds as follows :

Since $\frac{13m - 4}{17}$ is integral, $\frac{4(13m - 4)}{17}$ is also integral.

But $\frac{52m - 16}{17} = 3m + \frac{m - 16}{17}$; and $\frac{m - 16}{17}$ is integral.

Let $\frac{m - 16}{17} = n$; then

$$m = 17n + 16. \tag{6}$$

Substituting in (4) the value of m in (6), gives

$$x = 221n + 215. \tag{7}$$

Substituting in (3) the value of x in (7), gives

$$\frac{221n + 202}{23}. \tag{8}$$

But $\frac{221n + 202}{23} = 9n + 8 + \frac{2(7n + 9)}{23}$.

Since (8) is integral,

$\frac{13(7n + 9)}{23}$ or $\frac{92n + 115 - n + 2}{23}$ is integral.

Hence, $\frac{-n + 2}{23}$ is integral.

Let $\frac{-n + 2}{23} = r$; then

$$n = 2 - 23r. \tag{9}$$

Substituting in (7) the value of n in (9), gives

$$x = 657 - 5083r. \qquad (10)$$

Making $r = 0$, gives $x = 657$, the required least number.

By taking $m = -1, -2, -3$, etc., other numbers may be found answering the conditions.

IV. Given $\left.\begin{matrix} x - 2y + z = 5, & (1) \\ 2x + y - z = 7, & (2) \end{matrix}\right\}$;

to find the least positive integral roots.

Eliminating z, $\qquad 3x - y = 12. \qquad (3)$

Whence $x = \dfrac{12 + y}{3} = 4 + \dfrac{y}{3}$; so that $\dfrac{y}{3}$ is integral.

Let $\dfrac{y}{3} = n$; then $\qquad y = 3n. \qquad (4)$

Substituting in (3) the value of y in (4),

$$x = n + 4. \qquad (5)$$

Substituting in (1) the values of x and y in (4) and (5),

$$z = 5n + 1. \qquad (6)$$

Putting $n = 1$ in (4), (5), and (6), gives

$x = 5$, $y = 3$, $z = 6$, the required least roots.

Putting $n = 2$, $\quad x = 6$, $y = 6$, $z = 11$.

" $n = 3$, $\quad x = 7$, $y = 9$, $z = 16$.

There are, therefore, an infinite number of integral solutions.

433. ***Find one or more sets of positive integral roots:***

1.	$2x + 3y = 9.$	*9.*	$19x - 14y = 11.$
2.	$4x + 29y = 150.$	*10.*	$17x - 7y = 1.$
3.	$3x + 29y = 151.$	*11.*	$23x - 9y = 929.$
4.	$7x + 15y = 225.$	*12.*	$8x - 23y = 19.$
5.	$3x + 4y = 39.$	*13.*	$39x - 56y = 11.$
6.	$8x + 13y = 500.$	*14.*	$3x + 5y = 73.$
7.	$7x + 13y = 405.$	*15.*	$5x + 9y = 37.$
8.	$2x + 7y = 125.$	*16.*	$8x + 11y = 49.$

17. $2x + 5y + 3z = 51$; $10x + 3y + 2z = 120.$

18. $3x + 5y + 7z = 560$; $9x + 25y + 49z = 2920.$

19. $2x + 11y - 3z = 35$; $3x - 2y + 3z = 16.$

PROBLEMS LEADING TO INDETERMINATE EQUATIONS

434. *1.* A farmer paid \$131 for some calves at \$11 a he and some hogs at \$13 a head. How many of each kind we there?

2. A gentleman distributed \$5 among some poor peopl giving \$.70 each to some and \$.30 each to the rest. H many were there?

3. I wish to purchase 100 animals for \$100; hogs at sheep at \$3, and lambs at \$.50 a head. How many of ea kind must I buy?

4. How many pairs of fractions are there with denomin tors 12 and 18 whose sum is $\frac{5}{6}$?

5. What is the least number which, when divided by 5, and 9, the remainders are 6, 7, and 8, respectively?

6. A company of men and women spend \$1000; the men paying \$19, and the women, \$13, each. How many in the company?

7. The number 200 may be divided into two parts, such that when the first is divided by 6 and the second by 11, the respective remainders are 5 and 4. What are the parts?

8. Find the least number which, being divided by 28, 19, and 15, leaves remainders 13, 2, and 7, respectively.

9. A number consists of three digits whose sum is 20; if 16 be taken from the number, and the remainder divided by 2, the digits will be reversed. What is the number?

10. A man buys horses and oxen for \$2500, paying \$113 for each horse and \$23 for each ox. How many of each kind did he buy?

11. Divide 30 into three parts, such that 7 times the first, 19 times the second, and 38 times the third, are together equal to 745.

12. Thirty persons, men, women, and children, spent together \$58. The men spent \$3$\frac{1}{2}$, the women \$1$\frac{3}{4}$, and the children \$$\frac{1}{4}$ each. How many were there of each?

13. The number of pages in a certain book divided by 11 leaves a remainder 7; divided by 7 leaves a remainder 3; and divided by 5 leaves a remainder 1. What is the least number of pages it can contain?

14. A bill of \$4.19 is paid with three kinds of coin, 25-cent pieces, 10-cent pieces, and 3-cent pieces. How many of each must be taken?

15. A, B, and C's fortunes are in the proportion of 5, 7, and 9. If they buy land at \$11, \$13, and \$15 an acre, respectively, they will have left, after each has paid for the greatest whole number of acres possible, \$1, \$2, and \$3, respectively. What is the least sum each can have?

PERMUTATIONS, COMBINATIONS, AND ARRANGEMENTS.

435. Permutations are the results obtained by grouping several objects in every possible order of succession, all the objects occurring in each group.

Thus, the permutations of 4 letters, *a*, *b*, *c*, and *d*, are:

Beginning with *a*,		*abcd, abdc, acbd, acdb, adbc, adcb.*
" " *b*,		*bacd, badc, bcad, bcda, bdac, bdca.*
" " *c*,		*cabd, cadb, cbad, cbda, cdab, cdba.*
" " *d*,		*dabc, dacb, dbac, dbca, dcab, dcba.*

436. Combinations are the different groups that can be made of n objects taken r together, n being greater than r, and no two groups containing the same objects.

Thus, the combinations of 5 letters, *a*, *b*, *c*, *d*, and *e*, taken 2 together, and then 3 together, are:

Beginning with *a*,		*ab, ac, ad, ae.*
" " *b*,		*bc, bd, be.*
" " *c*,		*cd, ce.*
" " *d*,		*de.*

Beginning with *a*,		*abc, abd, abe, acd, ace, ade.*
" " *b*,		*bcd, bce, bde.*
" " *c*,		*cde.*

437. Arrangements are permutations of combinations.

Thus, the arrangements of 4 letters, *a*, *b*, *c*, and *d*, taken 3 in a set, are made from their combinations as follows:

Combinations,	*abc,*	*abd,*	*acd,*	*bcd.*
Arrangements,	*abc,*	*abd,*	*acd,*	*bcd.*
	acb,	*adb,*	*adc,*	*bdc.*
	bac,	*bad,*	*cad,*	*cbd.*
	bca,	*bda,*	*cda,*	*cdb.*
	cab,	*cad,*	*dac,*	*dbc.*
	cba,	*cda,*	*dca,*	*dcb.*

438. To find the number of arrangements of m things taken n together.

1. Suppose that m letters, a, b, c, d, etc., are taken 2 together. If a is placed first, there are $m - 1$ other letters, each of which may go with a, giving ab, ac, ad, etc., making $m - 1$ arrangements. If b is placed first, the $m - 1$ other letters each placed with it will give ba, bc, bd, etc., to $m - 1$ arrangements. Hence, taking each of the m letters, each giving $m - 1$ arrangements, makes $m(m - 1)$ arrangements in all.

2. When 3 letters are required in each set, each of the $m(m - 1)$ arrangements with 2 in a set may be combined with each of the $m - 2$ letters remaining, giving for each of the $m - 2$ letters $m - 2$ arrangements, or for all

$$m(m - 1)(m - 2)$$

arrangements with 3 in a set.

3. Each of these sets, in like manner, gives with each of the $m - 3$ letters remaining $m - 3$ arrangements, or for all

$$m(m - 1)(m - 2)(m - 3)$$

arrangements with 4 in a set.

4. Finally, when the letters are taken n in a set, the number of arrangements must be $m(m-1)(m-2)(m-3)$, and so on to $m-(n-1)$ arrangements, or

$$m(m-1)(m-2)(m-3)\ldots.(m-n+1)$$

arrangements. Hence,

The number of arrangements of m things taken n in a set is equal to the continued product of the consecutive numbers from m down to $m-n+1$.

439. To find the number of permutations of m things.

In the case of permutations, m becomes equal to n. Hence the preceding expression for the number of arrangements becomes $m(m-1)(m-2)(m-3)\ldots.(m-m+1)$, or, reversing the expression, $1\times2\times3\times4\times\ldots.\times m$. Hence,

The number of permutations of m things is equal to the continued product of the consecutive numbers from 1 *to* m.

440. To find the number of combinations of m things taken n together.

It has been shown that with m things taken n together, the number of

$$Arrangements = m(m-1)(m-2)(m-3)\ldots.(m-n+1).$$

But the *arrangements* of m things taken n together are obtained by permuting the *combinations* of m things taken n together, each combination yielding $1\times2\times3\ldots.\times n$ arrangements. (Art. **437.**) Hence, if the number of arrangements of m things taken n together be divided by the number

of permutations of n things, it will give the number of combinations of m things taken n together. That is, of m things taken n together, the number of

$$\textit{Combinations} = \frac{m(m-1)(m-2)(m-3)\ldots.(m-n+1)}{1\times2\times3\times\ldots.\times n}.$$

Hence,

The number of combinations of m things taken n together is equal to the continued product of the consecutive numbers from m to m − n + 1, divided by the continued product of the consecutive numbers from 1 *to n.*

EXERCISES.

441. *1.* How many permutations can be made of the letters in the word *forest?* In the word *logarithms?* In the word *mucilage?*

2. How many arrangements and how many combinations can be made of 10 things taken 3 together? Taken 4 together? Taken 5 together?

3. In how many ways can 12 pupils be seated on a bench, so that they shall never, when taken in the same direction, be seated twice in the same order?

4. In how many different groups of 7 can 15 persons be placed, no two groups to contain the same persons?

5. In how many ways can the seven colors of the rainbow be made to succeed one another?

6. How many arrangements can be made of the letters in the word *Richmond,* taken 4 letters in a set? How many combinations?

7. How many different groups of 12 soldiers can be made of 25 soldiers?

8. In how many orders of succession can 10 pupils leave the class-room when dismissed?

INDETERMINATE CO-EFFICIENTS.

442. Indeterminate co-efficients are co-efficients assumed in a discussion. Their values are at first unknown, but may be found as the discussion proceeds.

443. The method of indeterminate co-efficients depends upon the following principles:

PRIN. I.—*If* $A + Bx + Cx^2 + Dx^3 + \text{etc.} = A' + B'x + C'x^2 + D'x^3 + \text{etc.}$, *then will the co-efficients of like powers of* x *be equal to each other.*

1. For since the equation is true for any value of x, it is true when $x = 0$. Making $x = 0$, the equation becomes $A = A'$. (These are the co-efficients of x^0.)

2. Dropping these equals, A and A', from the equation, it becomes $Bx + Cx^2 + Dx^3 + \text{etc.} = B'x + C'x^2 + D'x^3 + \text{etc.}$ Dividing by x, and then making $x = 0$, gives $B = B'$.

3. Again, dropping B and B', dividing by x and making $x = 0$, gives $C = C'$. In this manner we proceed for the remaining co-efficients.

PRIN. II.—*If* $A + Bx + Cx^2 + Dx^3 + \text{etc.} = 0$ *is true for all values of* x, *then will* A, B, C, D, etc., *be each equal to zero.*

The equation may be written $A + Bx + Cx^2 + Dx^3 + \text{etc.} = 0 + 0x + 0x^2 + 0x^3 + \text{etc.}$ Whence, from Principle I., it follows that $A = 0$, $B = 0$, etc.

EXAMPLES.

444. *1.* Develop $\frac{1-x}{1+x+x^2}$ into a series by the method of indeterminate co-efficients.

Assume $\frac{1-x}{1+x+x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$

Clearing of fractions,

$$1 - x = A + \left\{\begin{matrix} B \\ A \end{matrix}\right\} x + \left\{\begin{matrix} C \\ B \\ A \end{matrix}\right\} x^2 + \left\{\begin{matrix} D \\ C \\ B \end{matrix}\right\} x^3 + \left\{\begin{matrix} E \\ D \\ C \end{matrix}\right\} x^4 + \left\{\begin{matrix} F \\ E \\ D \end{matrix}\right\} x^5 + \text{etc.}$$

Whence, by Prin. I.,

$$A = 1, \qquad A + B = -1, \qquad A + B + C = 0,$$

$$B + C + D = 0, \qquad C + D + E = 0.$$

Solving these, $A = 1$, $B = -2$, $C = 1$, $D = 1$, and $E = -2$.

Substituting in the assumed equation,

$$\frac{1-x}{1+x+x^2} = 1 - 2x + x^2 + x^3 - 2x^4 + \text{etc.}$$

Actual division will give the same result.

2. Develop $\frac{1+2x}{1-3x^2}$ into a series.

In this method, since the series $A + Bx + Cx^2 + \text{etc.}$ is hypothetical, we must carefully observe whether the subsequent

processes develop any inconsistency. For example, perhaps a particular expression will not develop in the form assumed. If so, some inconsistency will appear in the process. Thus, were we to attempt to develop $\frac{2}{x^2 - x^3}$ by assuming $\frac{2}{x^2 - x^3} = A + Bx + Cx^2 + Dx^3 + \text{etc.}$, we should find, after clearing of fractions, that the first member had only the term 2, which is $2x^0$; and as there would be no corresponding term in the second member, we should have to write $2 = 0$, which is absurd. In general, when we equate the co-efficients, the assumed member must have a term containing as low a power of x as the lowest in the first member. This may be secured either by putting the expression to be developed into a proper form before assuming the series, or by assuming a series of proper form. Thus, for $\frac{2}{x^2 - x^3}$, we may write

$$\frac{1}{x^2} \cdot \frac{2}{1 - x},$$

and then assume

$$\frac{2}{1 - x} = A + Bx + Cx^2 + Dx^3 + \text{etc.},$$

and finally multiply by $\frac{1}{x^2}$; or it may be developed by assuming

$$\frac{2}{x^2 - x^3} = Ax^{-2} + Bx^{-1} + Cx^0 + Dx + Ex^2 + \text{etc.}$$

3. Develop $\frac{1 - x}{2x^2 + 3x^3}$ into a series.

4. Expand $\frac{1}{1 - 2x + x^2}$ into a series.

5. Expand $\frac{2}{3x^2 - 2x^3}$ into a series.

DIFFERENTIATION.

445. In certain classes of problems and discussions, the quantities involved are distinguished as *constant* and *variable.*

446. A **constant** quantity is one which maintains the same value throughout the same discussion, and is represented by one of the leading letters of the alphabet.

447. Variable quantities are such as may assume in the same discussion any value within certain limits determined by the nature of the problem, and are represented by the final letters of the alphabet.

If x is the radius of a circle and y is its area, $y = \pi x^2$, as we learn from Geometry, π being about 3.1416. Now if x, the radius, varies, y, the area, will vary; but π remains the same for all values of x and y. In this case, x and y are the variables, and π is a constant.

Again, if y is the distance a body falls in a time x, it is evident that x increases or decreases with y, *i. e.*, that when x varies y varies. We learn from Physics that $y = 16\frac{1}{12}x^2$, for comparatively small distances above the surface of the earth. In the expression $y = 16\frac{1}{12}x^2$, x and y are the variables, and $16\frac{1}{12}$ is a constant. $\therefore y \propto x^2$, read, *y varies as x.*

Once more, suppose we have $y^2 = 25x^3 - 3x^2 - 5$, as an expressed relation between x and y, and that this is the *only* relation required to exist between them; it is evident that we may give values to x *at pleasure,* and thus obtain corresponding values for y. Thus, if $x = 1$, $y = \pm\sqrt{17}$; if $x = 2$, $y = \pm\sqrt{183}$, etc., etc. In such a case, x and y

are called variables. But we notice that if we give to x such a value as to make $3x^2 + 5 > 25x^3$ (as, for example, $\frac{1}{2}$, $\frac{1}{4}$, etc.), y will be imaginary. This is the kind of limitation referred to in our definition of variables.

448. A **function** is a quantity, or a mathematical expression, conceived as depending for its value upon some other quantity or quantities.

A man's wages *for a given time* is a function of the amount received per day, or, in general, his wages is a function of the time he works and the amount he receives per day. In the expression $y = 16\frac{1}{12}x^2$ (**447**), y is a function of x, *i. e.*, the space fallen through is a function of the time. The expression $2ax^2 - 3x + 5b$, or any expression containing x, may be spoken of as a function of x.

449. To indicate that one variable, as y, is a function of another, as x, and without being more specific, we write $y = f(x)$, read "y is a function of x." This means nothing more than that y is equal to some expression containing the variable x and any constants. Several *different* expressions, each of which contains x, are written $f(x)$, $\phi(x)$, or $f'(x)$, etc., and read "the f function of x," "the ϕ function of x," or "the f' function of x."

The expression $f(x)$ may stand for $x^3 - 2x + 5$, or for $3(a^2 - x^2)$, or for *any* expression containing x combined in any way with itself or with constants. But in the same discussion $f(x)$ will mean the same thing throughout.

450. In equations expressing the relation between two variables, as in $y = 3ax^3 - x^2$, it is customary to speak of one of the variables, as y, as a function of the other, x.

Moreover, it is convenient to think of x as varying and thus producing change in y. When so considered, x is called the *independent* and y the *dependent* variable. Or we may speak of y as a function of the variable x.

451. An **infinitesimal** is a quantity conceived under such a form, or law, as to be necessarily less than any assignable quantity.

Infinitesimals are the increments by which continuous number or quantity may be *conceived* to change value, or grow.

Time affords a good illustration of continuous quantity, or number. Thus a period of time, as 5 hours, increases, or grows, to another period, as 7 hours, by infinitesimal increments, *i. e.*, not by hours, minutes, or even seconds, but by elements which are less than *any* assignable quantity.

452. Consecutive values of a variable are values that differ from each other by less than any assignable quantity, *i. e.*, by an infinitesimal. Consecutive values of a *function* are values which correspond to consecutive values of its variable.

453. A **differential** of a function, or variable, is the difference between two consecutive states of the function, or variable. It is the same as an infinitesimal.

Resuming the illustration $y = 16\frac{1}{12}x^2$, let x be thought of as some particular period of time (as 5 seconds), and y as the distance through which the body falls in that time. Also, let x' represent a period of time only infinitesimally greater than x, and y' the distance through which the body falls in time x'. Then x and x' are consecutive values of x, and y

and y' are consecutive values of y. Again, the difference between x and x', as $x' - x$, is a differential of the variable x, and $y' - y$ is a differential of the function y.

454. Notation.—A differential of x is represented by writing the letter d before x, thus dx. Also, dy means, and is read, "differential y."

455. To **differentiate** a function is to find an expression for the increment of the function due to an infinitesimal increment of the variable; or it is the process of finding the relation between the infinitesimal increment of the variable and the corresponding increment of the function.

RULES FOR DIFFERENTIATING.

456. Rule I.—*To differentiate a single variable, simply write the letter d before it.*

This is merely doing what the notation requires. Thus, if x and x' are consecutive states of the variable x, *i. e.*, if x' is what x becomes when it has taken an infinitesimal increment, $x' - x$ is the differential of x, and is to be written dx. In like manner, $y' - y$ is to be written dy, y' and y being consecutive values.

457. Rule II.—*Constant factors or divisors appear in the differential the same as in the function.*

Let us take the function $y = ax$, in which a is any constant, integral or fractional. Let x take an infinitesimal increment dx, becoming $x + dx$; and let dy be the corresponding increment of y, so that when x becomes $x + dx$, y becomes $y + dy$. We then have

1st state of the function, $y = ax$;
2d, or consecutive state, $y + dy = a(x + dx) = ax + adx.$
Subtracting the 1st from the 2d, $dy = adx,$

which result being the difference between two consecutive states of the function, is its differential. Now a appears in the differential just as it was in the function. This would evidently be the same if a were a fraction, as $\frac{1}{m}$. We should then have, in like manner, $dy = \frac{1}{m}dx$, for the differential of

$$y = \frac{1}{m}x.$$

458. Rule III.—*Constant terms disappear in differentiating; or the differential of a constant is* 0.

Let us take the function $y = ax + b$, in which a and b are constant. Let x take an infinitesimal increment and become $x + dx$; and let dy be the increment which y takes in consequence of this change in x, so that when x becomes $x + dx$, y becomes $y + dy$. We then have,

1st state of the function, $y = ax + b.$
2d, or consecutive state, $y + dy = a(x + dx) + b = ax + adx + b.$
Subtracting the 1st from the 2d, $dy = adx,$

which being the difference between two consecutive states of the function, is its differential. Now from this differential the constant b has disappeared.

We may also say that as a constant retains the same value, there is no difference between its consecutive states (properly it has no consecutive states). Hence, the differential of a constant may be spoken of as 0.

459. Rule IV.—*To differentiate the algebraic sum of several variables, differentiate each term separately and connect the differentials with the same signs as the terms.*

Let $u = x + y - z$, u representing the algebraic sum of the variables x, y, and $-z$. Then is $du = dx + dy - dz$. For, let dx, dy, and dz be infinitesimal increments of x, y, and z; and let du be the increment which u takes in consequence of the infinitesimal changes in x, y, and z. We then have,

1st state of the function, $u = x + y - z$;
2d, or consecutive state, $u + du = x + dx + y + dy - (z + dz)$.
Or, $u + du = x + dx + y + dy - z - dz$.
Subtracting the 1st state from the 2d, $du = dx + dy - dz$.

460. Rule V.—*The differential of the product of two variables is the differential of the first into the second, plus the differential of the second into the first.*

Let $u = xy$ be the first state of the function. The consecutive state is $u + du = (x + dx)(y + dy) = xy + ydx + xdy + dx \cdot dy$. Subtracting the first state from the consecutive state, we have the differential, *i. e.*, $du = ydx + xdy + dx \cdot dy$. But, as $dx \cdot dy$ is the product of two infinitesimals, it is infinitely less than the other terms (ydx and xdy), and hence, having no value as compared with them, is to be dropped. Therefore, $du = ydx + xdy$.

461. Rule VI.—*The differential of the product of several variables is the sum of the products of the differential of each into the product of all the others.*

Let $u = xyz$; then $du = yzdx + xzdy + xydz$. For the

1st state of the function is $u = xyz$, and the 2d, or consecutive state, $u + du = (x + dx)(y + dy)(z + dz)$, or $u + du = xyz + yzdx + xzdy + xydz + xdydz + ydxdz + zdxdy + dxdydz$. Subtracting, and dropping all infinitesimals of infinitesimals, we have $du = yzdx + xzdy + xydz$.

In a similar manner, the rule can be demonstrated for any number of variables.

462. Rule VII.—*The differential of a fraction having a variable numerator and denominator is the differential of the numerator multiplied by the denominator, minus the differential of the denominator multiplied by the numerator, divided by the square of the denominator.*

Let $u = \frac{x}{y}$; then is $du = \frac{ydx - xdy}{y^2}$. For, clearing of fractions, $yu = x$. Differentiating this by Rule V., we have $udy + ydu = dx$. Substituting for u its value, $\frac{x}{y}$, this becomes $\frac{xdy}{y} + ydu = dx$. Finding the value of du, we have $du = \frac{ydx - xdy}{y^2}$. Hence,

463. *The differential of a fraction having a constant numerator and a variable denominator is the product of the numerator with its sign changed into the differential of the denominator, divided by the square of the denominator.*

Let $u = \frac{a}{y}$. Differentiating this by the rule and calling the differential of the constant (a) 0, we have

$$du = \frac{y0 - ady}{y^2} = \frac{-ady}{y^2}.$$

464. Rule VIII.—*The differential of a variable affected with an exponent is the continued product of the exponent, the variable with its exponent diminished by 1, and the differential of the variable.*

1st. *When the exponent is a positive integer.* Let $y = x^m$, m being a positive integer; then $dy = mx^{m-1}dx$. For $y = x^m = x \cdot x \cdot x \cdot x \ldots$. to m factors. Now, differentiating this by Rule VI., we have $dy = (xxx \ldots$. to $m-1$ factors$)dx + (xxx \ldots$. to $m-1$ factors$)dx +$ etc., to m terms; or $dy = x^{m-1}dx + x^{m-1}dx + x^{m-1}dx +$ etc., to m terms. Therefore, $dy = mx^{m-1}dx$.

2d. *When the exponent is a positive fraction.* Let $y = x^{\frac{m}{n}}$, $\frac{m}{n}$ being a positive fraction; then $dy = \frac{m}{n}x^{\frac{m}{n}-1}dx$. For involving both members to the n^{th} power, we have $y^n = x^m$. Differentiating this as just shown, we have $ny^{n-1}dy = mx^{m-1}dx$. Now, from $y = x^{\frac{m}{n}}$, we have $y^{n-1} = x^{\frac{mn-m}{n}}$. Substituting this in the last, it becomes $nx^{\frac{mn-m}{n}}dy = mx^{m-1}dx$; whence, $dy = \frac{m}{n}x^{m-1-\frac{mn-m}{n}}dx = \frac{m}{n}x^{\frac{m}{n}-1}dx$.

3d. *When the exponent is negative.* Let $y = x^{-n}$, n being integral or fractional; then $dy = -nx^{-n-1}dx$. For $y = x^{-n} = \frac{1}{x^n}$, which, differentiated by Rule VII., gives

$$dy = -\frac{nx^{n-1}dx}{x^{2n}} = -nx^{-n-1}dx.$$

It is desirable that the pupil not only become expert in writing out differentials, but that he know what the operation signifies. Thus, in the equation $y = 5x$, if x changes value,

y must change also, in order to keep the equation true. In this simple case, it is easy to see that y must change 5 times as fast as x. This is what differentiation shows.

Again, let $y = 3x^2 - 2x + 4$, and let it be required to find *the relative rate of change of x and y*. Differentiating, we have $dy = 6xdx - 2dx = (6x - 2)\,dx$. This shows that, if x takes an infinitesimal increment represented by dx, y takes one (represented by dy) which is $6x - 2$ times as large; *i. e.*, that y increases $6x - 2$ times as fast as x.

EXAMPLES.

465. *1.* Differentiate $y = 3x^2 - 2x + 4$.

The result is $dy = 6xdx - 2dx$. Which is thus obtained: By Rule I., the differential of y is dy. To differentiate the second member, we differentiate each term separately, according to Rule IV. In differentiating $3x^2$, the factor 3 is retained in the differential, Rule II., and the differential of x^2 is, by Rule VIII., $2xdx$. Hence, the differential of $3x^2$ is $6xdx$. The differential of $-2x$ is $-2dx$. By Rule III., the constant 4 disappears from the differential, or its differential is 0.

2. Differentiate $y = 2ax^2 + 4ax^3 - x + m$.

3. Differentiate $y = 5bx^3 - 30x^3 + 4x$.

4. Differentiate $y = Ax^2 + Bx^3 + Cx^4$.

5. Differentiate $y = x^5 - x^3$, and explain the significance of the result.

6. In order to keep the relation $2y = 3x^2$ true as x varies, how must y vary in relation to x? What is the relative rate of change when $x = 4$? When $x = 2$? When $x = 1$? When $x = \frac{1}{3}$? When $x = \frac{1}{6}$?

7 to *12*. Differentiate the following: $u = \frac{2x^2}{3y}$; $u = \frac{x^2-1}{x^2+1}$; $y = x^{\frac{1}{2}}z^2$; $u = x^2y^3 + 6x$; $y = x^5 - 3x^4 + 4x^3 - x^2 + 1$; and $y = \frac{1}{2}x^2 - \frac{2}{3}x^3 + x$.

13 to *17*. Differentiate $y = (a^2 + x^3)^5$; $y = (a + x^2)^{\frac{3}{2}}$; $y = (3x - 2)^{\frac{2}{3}}$; $y = (2 - x^2)^{-2}$; and $y = (1 + x)^{-\frac{1}{2}}$.

Such examples should be solved by considering the entire quantity within the parenthesis as the variable. This is evidently admissible, since any expression that contains a variable is variable when taken as a whole. Thus, to differentiate $y = (a + x^2)^{\frac{3}{5}}$, take the continued product of the exponent ($\frac{3}{5}$), the variable $(a + x^2)$ with its exponent diminished by 1 [*i. e.*, $(a + x^2)^{-\frac{1}{5}}$], and the differential of the variable (*i. e.*, the differential of $a + x^2$, which is $2xdx$). This gives $dy = \frac{3}{5}(a + x^2)^{-\frac{1}{5}} 2xdx$, or

$$dy = \tfrac{6}{5}x(a + x^2)^{-\frac{1}{5}}dx = \frac{4xdx}{3\sqrt[3]{a + x^2}}.$$

18 to *22*. Differentiate $\frac{1}{1+x}$; $\frac{1}{(1+x)^2}$; $\frac{1}{(1+x)^3}$; $-m\frac{1}{(1+x)^5}$; and $-m\frac{1}{(1+x)^8}$.

23. In the expression $6x^3$, when x is greater than 1 does the function $(6x^3)$ change faster or slower than x? How when x is less than $\frac{1}{2}$? What does the process of differentiating $6x^3$ signify?

BINOMIAL THEOREM.

466. For brevity of notation, the continued product of all the consecutive numbers from 1 to n is indicated thus, $\lfloor n$, and is read, *factorial n*.

Thus, $1 \times 2 \times 3 \times 4$ is written $\lfloor 4$, and is read *factorial* 4.

467. *Whatever the value of m, i. e., whether integral or fractional, positive or negative,*

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{2}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{\lfloor 3}a^{m-3}x^3 + \frac{m(m-1)(m-2)(m-3)}{\lfloor 4}a^{m-4}x^4 + \text{etc}$$

Assume

$$(a+x)^m = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}, \quad (1)$$

in which A, B, C, etc., are indeterminate co-efficients independent of x (*i. e.*, constants), and are to be determined. To determine these co-efficients, we proceed as follows:

Differentiating (1), we have

$$m(a+x)^{m-1}dx = Bdx + 2Cxdx + 3Dx^2dx + 4Ex^3dx + Fx^4dx + \text{etc.}$$

Dividing by dx, we have

$$m(a+x)^{m-1} = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \text{etc.} \quad (2)$$

Differentiating (2) and dividing by dx, we have

$$m(m-1)(a+x)^{m-2} = 2C + 2\cdot 3Dx + 3\cdot 4Ex^2 + 4\cdot 5Fx^3 + \text{etc.} \quad (3)$$

Differentiating (3) and dividing by dx, we have

$$m(m-1)(m-2)(a+x)^{m-3} = 2\cdot 3D + 2\cdot 3\cdot 4Ex + 3\cdot 4\cdot 5Fx^2 + \text{etc.} \quad (4)$$

Differentiating (4) and dividing by dx, we have

$$m(m-1)(m-2)(m-3)(a+x)^{m-4} = 2\cdot 3\cdot 4\cdot 5F + \text{etc.} \quad (5)$$

We have now gone far enough to enable us to determine the co-efficients A, B, C, D, E, etc., and doubtless also the law of the series.

The foregoing equations are to be true for all values of x, and the co-efficients A, B, C, etc., are constants, *i. e.*, have the same values for one value of x as for another, so that if their values can be found for one value of x, these will be their values in all cases. Now, making $x = 0$, we have from (1),

$$A = a^m;$$

from (2), $$B = ma^{m-1};$$

from (3), $$C = \frac{m(m-1)}{\lfloor 2}a^{m-2};$$

from (4), $$D = \frac{m(m-1)(m-2)}{\lfloor 3}a^{m-3};$$

from (5), $$E = \frac{m(m-1)(m-2)(m-3)}{\lfloor 4}a^{m-4}.$$

These values substituted in (1) give

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{\lfloor 2}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{\lfloor 3}a^{m-3}x^3 + \frac{m(m-1)(m-2)(m-3)}{\lfloor 4}a^{m-4}x^4 + \text{etc.},$$

which is the *Binomial Formula.*

468. *The n^{th} or general term of the series is*

$$\frac{m(m-1)(m-2)\ldots(m-n+2)}{\lfloor n-1}a^{m-n+1}x^{n-1}.$$

For we observe that the last factor in the numerator of the co-efficient of any particular term is m — the number of the term less 2, *i. e.*, for the n^{th} term, $m-(n-2)$, or $m-n+2$; and the last factor in the denominator is the number of the term — 1, *i. e.*, for the n^{th} term, $n-1$. The exponent of x in any particular term is m — the number of the term less 1, *i. e.*, for the n^{th} term, $m-(n-1)$, or $m-n+1$; and the exponent of y in any term is one less than the number of the term, *i. e.*, for the n^{th} term, $n-1$.

469. In a series, the **scale of relation** is the relation that exists between any term or set of terms and the next term or set of terms.

470. *The scale of relation in the binomial series is* $\left(\frac{m+1}{n}-1\right)\frac{y}{x}$, *since the n^{th} term multiplied by this produces the $(n+1)^{th}$ term.*

This is readily seen by inspecting the series, or by writing the $(n+1)^{th}$ term and dividing it by the n^{th}. Thus, substituting in the general term as given above, $n+1$ for n, we have

$$\frac{m\,(m-1)\,(m-2)\,\ldots\,(m-n+1)}{\lfloor n}\,x^{m-n}y^n,$$

as the $(n+1)^{th}$ term. This divided by the n^{th}, or preceding term, gives $\left(\frac{m-n+1}{n}\right)\frac{y}{x}$, or $\left(\frac{m+1}{n}-1\right)\frac{y}{x}$.

EXAMPLES.

471. *1.* Expand $(1+x)^{\frac{1}{2}}$ to five terms.

2. Expand $(1+2x)^{\frac{1}{2}}$ to five terms.

3. Expand $(1-3x)^{\frac{2}{3}}$ to five terms.

4. Expand $(1+x)^{-2}$ to five terms.

5. Expand $(1-2x)^{-3}$ to five terms.

6. Expand $(1-x^2)^{\frac{1}{2}}$ to five terms.

7. Expand $(1-3x)^{-1}$ to five terms.

8. Expand $(a+x)^{-4}$ to five terms.

9. Expand $\left(1+\frac{x}{3}\right)^{-3}$ to five terms.

10. Expand $(1-x^2)^{\frac{7}{2}}$ to five terms.

11. Expand $\frac{1}{(a^2+x^2)^2}$ to five terms.

12. Expand $(a^2-x^2)^{\frac{1}{2}}$ to five terms.

13. Expand $(a^{\frac{3}{2}}-3b^{\frac{1}{2}}x)^{\frac{2}{5}}$ to five terms.

LOGARITHMS.

472. A **logarithm** is the exponent by which a fixed number is to be affected in order to produce any required number. The fixed number is called the **base of the system.**

Thus, let the base be 3: then the logarithm of 9 is 2; of 27, 3; of 81, 4; of 19683, 9; for $3^2 = 9$; $3^3 = 27$; $3^4 = 81$; and $3^9 = 19683$. Again, if 64 is the base, the logarithm of 8 is $\frac{1}{2}$, or .5, since $64^{\frac{1}{2}}$, or $64^{.5} = 8$; and $\frac{1}{3}$, or .333+, is the logarithm of 4, since $64^{\frac{1}{3}}$, or $64^{.333+} = 4$. Once more, since $64^{\frac{2}{3}}$, or $64^{.666+} = 16$, $\frac{2}{3}$, or .666+, is the logarithm of 16, if the base is 64. Finally, $64^{-\frac{1}{2}}$, or $64^{-.5} = \frac{1}{8}$, or .125; hence, $-\frac{1}{2}$, or $-.5$, is the logarithm of $\frac{1}{8}$, or .125, when the base is 64. In like manner, with the same base, $-\frac{1}{3}$, or $-.333+$, is the logarithm of $\frac{1}{4}$, or .25.

EXAMPLES.

473. *1.* If 2 is the base, what is the logarithm of 4? of 8? of 32? of 128? of 1024?

2. If 5 is the base, what is the logarithm of 625? Of 15625? Of 125? Of 25?

3. If 10 is the base, what is the logarithm of 100? Of 1000? Of 10000? Of 10000000?

4. If 2 is the base, what is the logarithm of $\frac{1}{4}$, or .25? Of $\frac{1}{8}$, or .125? Of $\frac{1}{32}$, or .03125?

5. If 8 is the base, of what number is $\frac{2}{3}$, or .666+, the logarithm? Of what number is $\frac{1}{3}$, or 1.333+, the logarithm? Of what number is 2 the logarithm?

474. A **system of logarithms** is a scheme by which all numbers can be represented, either exactly or approximately, by affecting a fixed number (the base) with exponents. Negative numbers can have no logarithms.

475. There are two systems of logarithms in common use, called, respectively, the **Briggean** or **common** system, and the **Napierian** or **hyperbolic** system. The base of the former is 10, and of the latter 2.71828+.

476. One of the most important uses of logarithms is to facilitate the multiplication, division, involution, and the extraction of roots of large numbers. These processes are performed upon the following principles:

477. PRIN. I.—*The sum of the logarithms of two numbers is the logarithm of their product.*

Let a be the base of the system. Let m and n be any two numbers whose logarithms are x and y respectively. Then, by definition, $a^x = m$, and $a^y = n$. Multiplying these equations together, we have $a^{x+y} = mn$ Whence, $x + y$ is the logarithm of mn.

478. PRIN. II.—*The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

Let a be the base of the system, and m and n any two numbers whose logarithms are, respectively, x and y. Then, by definition, we have $a^x = m$, and $a^y = n$. Dividing, we have $a^{x-y} = \frac{m}{n}$. Whence, $x - y$ is the logarithm of $\frac{m}{n}$.

479. PRIN. III.—*The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

Let a be the base, and x the logarithm of m. Then $a^x = m$; and raising both to any power, as the z^{th}, we have, $a^{xz} = m^z$. Whence, xz is the logarithm of the z^{th} power of m.

480. PRIN. IV.—*The logarithm of any root of a number is the logarithm of the number divided by the number expressing the degree of the root.*

Let a be the base, and x the logarithm of m. Then $a^x = m$. Extracting the z^{th} root, we have $a^{\frac{x}{z}} = \sqrt[z]{m}$. Whence, $\frac{x}{z}$ is the logarithm of $\sqrt[z]{m}$.

481. To apply these principles practically, a **table of logarithms** is needed.

482. To compute a table of logarithms, the base being 10.

It is evident that it is necessary to compute the logarithms of *prime* numbers only, since the logarithm of a composite number is the sum of the logarithms of its factors.

To compute the logarithms of the series of prime numbers. In the first place, we know that the logarithm of 1 is 0, since $10^0 = 1$. Also, the logarithm of 10 is 1, since $10^1 = 10$. Now if we find the logarithm of 5, we can get the logarithm of 2 by subtracting the logarithm of 5 from log. 10. If there be any number which is the logarithm of 5, it is evident it must lie between 0, which is log. 1, and 1, which is log. 10.

Therefore, starting with

$$10^0 = 1,$$

and

$$10^1 = 10.$$

Multiplying them together, we have $10^1 = 10.$

Extracting the square root,

$$10^{.5} = \sqrt{10}$$
$$= 3.162277+.$$

Again, as 5 lies between 10 and 3.162277+, its logarithm lies between 1 and .5. Multiplying the last two equations, we have

$$10^{1.5} = 31.62277+.$$

Extracting the square root,

$$10^{.75} = \sqrt{31.62277+}$$
$$= 5.623413+.$$

Again,

$$10^{.5} = 3.162277+,$$

and

$$10^{.75} = 5.623413+.$$

Multiplying,

$$10^{1.25} = 17.7827895914+.$$

Extracting the square root,

$$10^{.625} = \sqrt{17.7827895914+}$$
$$= 4.216964+.$$

Again, multiplying this last by $10^{.75} = 5.623413+$, as 5 lies between these numbers, and extracting the square root, we have $10^{.6875} = 4.869674+$. In each case the exponent of 10 is the logarithm of the number; thus, .6875 is log. 4.869674+. Continuing this process to 22 operations, we have log. 5.000000+ = .698970+, which is sufficiently accurate for ordinary purposes.

Now log. 10 — log. 5 = log. 2 = 1 — .698970 = .301030.

To find log. 3, we would take $10^{.5} = 3.162277+$, and $10^0 = 1$, and proceed as before.

Were it our purpose to find log. 11, the computation would be as follows:

$$10^1 = 10$$
$$10^2 = 100$$
$$\overline{10^3 = 1000.}$$

$$10^{1.5} = \sqrt{1000} = 31.62277+$$
$$10^1 = \quad 10$$
$$\overline{10^{2.5} = 316.2277+.}$$

$$10^{1.25} = \sqrt{316.2277+} = 17.78278+$$
$$10^1 = \quad 10$$
$$\overline{10^{2.25} = 177.8278+.}$$

$$10^{1.125} = \sqrt{177.8278+} = 13.33521+$$
$$10^1 = \quad 10$$
$$\overline{10^{2.125} = 133.3521+.}$$

$$10^{1.0625} = \sqrt{133.3521+} = 11.54782-$$
$$10^1 = \quad 10$$
$$\overline{10^{2.0625} = 115.4782+.}$$

$$10^{1.03125} = \sqrt{115.4782+} = 10.74607+$$
$$10^{1.0625} = \quad 11.54782-$$
$$\overline{10^{2.09375} = 124.09368+.}$$

$$10^{1.046875} = \sqrt{124.09368+} = 11.13973+$$
$$10^{1.03125} = \quad 10.74607+$$
$$\overline{10^{2.078125} = 119.70845+.}$$

$$10^{1.0390625} = 10.94113+.$$

Whence, 1.0390625 is log. 10.94113; and proceeding with the computation, the logarithm of 11 may be found with sufficient accuracy. In this manner a table of logarithms might be computed.

N.	0	1	2	3	4	5	6	7	8	9	D.
280	447158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	..95	154
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	.146	.296	.447	.597	.748	151
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	.116	.263	.410	.557	.704	.851	.998	1145	147
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	9958	..99	.239	.380	.520	.661	.801	.941	1081	1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	9687	9824	9962	..99	.236	.374	.511	.648	.785	.922	137
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	9203	9337	9471	9606	9740	9874	...9	.143	.227	.411	134
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
N.	0	1	2	3	4	5	6	7	8	9	D.

483. To find the logarithm of a number from the table.

Let it be required to find the logarithm of 325. Now, as the logarithm of 100 is 2, and of 1000 is 3, the logarithm of 325 must be between 2 and 3, *i. e.*, 2 and a fraction. The fractional part is all that is given in the table, as the integral can be known by simple inspection. Looking in the table down the column marked N (numbers), we find 325, and opposite it, in the column headed 0, we find 1883, but just above this we observe 51, which belongs to this logarithm and which is simply omitted to save space in the table, since it really belongs as a prefix to all the logarithms clear down to the number 332, where it is replaced by 52. Prefixing the 51 to the 1883, we have .511883 as the decimal part of the logarithm. Hence, log. 325 is 2.511883. In like manner, the logarithm *of any number represented by three figures* between 279 and 331 is found from the table.

To find the logarithm of a number represented by four figures. Let it be required to find the logarithm of 2936. Looking for 293 (the first three figures) in the column of numbers, and then passing to the right until reaching the column headed 6, the fourth figure, we find 7756, to which prefixing the figures 46, which belong to all the logarithms following them till some others are indicated, we have for the decimal part of the logarithm of 2936, .467756. But, as 3 is the logarithm of 1000, and 4 of 10000, log. 2936 is 3 and this decimal, or log. 2936 = 3.467756.

To find the logarithm of a number represented by more than four figures. Let it be required to find the logarithm of 2845672. Finding the decimal part of logarithm of the first

four figures, 2845, as before, we find it to be .454082. Now the logarithm of 2846 is 153 (millionths, really) more than than that of 2845. Hence, assuming that if an increase of the number by 1000 makes an increase in its logarithm of 153, an increase of 672 in the number will make an increase in the logarithm of $\frac{672}{1000}$ or .672 of 153, or 103. Omitting lower orders and adding this to .454082, we have .454185 as the decimal part of log. 2845672. The integral part is 6, since 2845672 lies between the 6th and 7th powers of 10. Hence, log. 2845672 = 6.454185.

NOTE 1.—If in seeking the logarithm of any number, any of the dots noticed in the table are passed, their places are to be filled with 0's, and the first two figures of the decimal of the logarithm taken from the 0 column in the line below. Thus, log. 3166 is 3.500511. This arrangement of the table is a mere matter of convenience to save space.

NOTE 2.—The column marked D is called the column of Tabular Differences; and any number in it is the difference between the logarithms found in columns 4 and 5, which is usually the same as between any two consecutive logarithms in the same horizontal line. The assumption made in using this difference, viz., that the logarithms increase in the same ratio as the numbers, is only approximately true, but still is accurate enough for ordinary use.

484. The integral part of a logarithm is called the **characteristic** and the decimal part the **mantissa.**

485. *The mantissa of a decimal fraction, or of a mixed number, is the same as the mantissa of the number considered as integral.*

It was found above that log. 2845672 = 6.454185. Now this means that $10^{6.454185} = 2845672$. Dividing by 10 successively, we have

$10^{5.454185} = 284567.2$, or log. $284567.2 = 5.454185$;
$10^{4.454185} = 28456.72$, or log. $28456.72 = 4.454185$;
$10^{3.454185} = 2845.672$, or log. $2845.672 = 3.454185$;
$10^{2.454185} = 284.5672$, or log. $284.5672 = 2.454185$;
$10^{1.454185} = 28.45672$, or log. $28.45672 = 1.454185$;
$10^{0.454185} = 2.845672$, or log. $2.845672 = 0.454185$.

Now, if we continue the operation of division, only writing $0.454185 - 1$, $\bar{1}.454185$, meaning by this that the characteristic is negative and the mantissa positive, and the subtraction not performed, we have

$10^{\bar{1}.454185} = .2845672$, or log. $.2845672 = \bar{1}.454185$,
$10^{\bar{2}.454185} = .02845672$, or log. $.02845672 = \bar{2}.454185$;
$10^{\bar{3}.454185} = .002845672$, or log. $.002845672 = \bar{3}.454185$;
etc., etc.

The characteristic of an integral number, or of a mixed integral number and decimal, is one less than the number of integral places, as will appear by comparing such numbers with the powers of 10. The characteristic of a number entirely decimal fractional is negative, and one greater than the number of 0's immediately following the decimal point, as appears from the last demonstration, or as appears from the fact that $10^{-1} = \frac{1}{10} = .1$; $10^{-2} = \frac{1}{100} = .01$; $10^{-3} = \frac{1}{1000} = .001$; etc.

EXAMPLES.

486. *1.* Find the logarithms of 285. Of 3145. Of 29056.

2. Of 30942. Of 298.026. Of 32.56.

3. Of 2.864. Of 3205. Of .00317. Of .00000328.

487. To find a number corresponding to a given logarithm.

Let it be required to find the number corresponding to the logarithm 5.515264. Looking in the table for the *next less* mantissa, we find .515211, the number corresponding to which is 3275 (no account now being taken as to whether it is integral, fractional or mixed; as in any case the figures will be the same). Now, from the *tabular difference*, in column D, we find that an increase of 133 (millionths, really) upon this logarithm (.515211) would make an increase of 1 in the number, making it 3276. But the given logarithm is only 53 greater than this, hence it is assumed (though only approximately correct) that the increase of the number is $\frac{53}{133}$ of 1, or $53 \div 133 = .3984+$. This added (the figures annexed) to 3275, gives 32753984+. The characteristic, being 5, indicates that the number lies between the 5th and 6th powers of 10, and hence has 6 integral places.

$$\therefore\ 5.515264 = \log.\ 327539.84+.$$

EXAMPLES.

488. *1.* Find the numbers corresponding to the following logarithms: 3.467521; $\bar{2}$.467521; 0.467521; 4.520281; 1.520281; $\bar{4}$.520281; 0.520281; $\bar{1}$.520281; 2.490160; $\bar{2}$.490160; and 0.490160.

2. If $28.035 : 3.2781 :: 3114.27 : x$, what logarithmic operations will find x?

Solution.—The logarithm of the product of the means is the sum of their logarithms; and the logarithm of the quotient of this product divided by the first extreme is the logarithm of said product minus the logarithm of the other extreme. $\therefore$ log. x = log. 3.2781 + log. 3114.27 −log. 28.035 = 0.515622 + 3.493356 − 1.447700 = 2.561278. Having a

table sufficiently extended, the number corresponding to this logarithm could be found, and would be the value of x.

3. Find the product of 23.14 by 5.062, knowing that log. 23.14 is 1.364363, log. 5.062 is 0.704322, and log. 117.1347 is 2.068685.

4. How is 287 raised to the 5th power by means of logarithms? How is the 5th root extracted?

5. Extract the 5th root of 31152784.1 by means of logarithms, knowing that log. 31152784.1 = 7.493497.

6. What is the cube root of 30?

7. What is the cube root of .03?

Solution.—Log. .03 = $\bar{2}.477121$. Now to divide this by 3, we have to bear in mind that the characteristic *alone* is negative, *i. e.*, $\bar{2}.477121 = -2+.477121$, or -1.522879. This divided by 3 gives $-.507626$, or $0-.507626 = \bar{1}.492374$. But a more convenient method of effecting this division is to write for the -2, $-3+1$, whence we have for $\bar{2}.477121$, $-3+1.477121$, which, divided by 3, gives $\bar{1}.492374$, nearly.

8. Divide $\bar{3}.261453$ by 2; by 4; by 5.

THE LOGARITHMIC SERIES.

489. The **modulus** of a system of logarithms is a constant factor which depends upon the base of the system and characterizes the system.

490. *The differential of the logarithm of a number is the differential of the number multiplied by the modulus of the system, divided by the number;*

Or, in the Napierian system, the modulus being 1, *the differential of the logarithm of a number is the differential of the number divided by the number.*

Let x represent *any* number, *i. e.*, be a variable, and n be a constant, such that $y = x^n$. Then $\log y = n \log x$. Differentiating $y = x^n$, we have $dy = nx^{n-1}dx$; whence,

$$n = \frac{dy}{x^{n-1}dx} = \frac{dy}{\frac{x^n}{x}dx} = \frac{dy}{\frac{y}{x}dx} = \frac{\frac{dy}{y}}{\frac{dx}{x}}. \qquad (1)$$

Again, whatever the differentials of $\log y$ and $\log x$ are, n being a constant factor we shall have the differential of $\log y$ equal to n times the differential of $\log x$, which may be written,

$$d(\log y) = n \cdot d(\log x),$$

whence,

$$n = \frac{d(\log y)}{d(\log x)}. \qquad (2)$$

Now, equating the values of n as represented in (1) and (2), we have

$$\frac{d(\log y)}{d(\log x)} = \frac{\frac{dy}{y}}{\frac{dx}{x}}.$$

Whence, $d(\log y)$ bears the same ratio to $\frac{dy}{y}$, as $d(\log x)$ does to $\frac{dx}{x}$. Let m be this ratio. Then

$$d(\log y) = \frac{mdy}{y}, \quad \text{and} \quad d(\log x) = \frac{mdx}{x}.$$

We are now to show that m is constant and depends on the base of the system.

To do this, take $y = z^{n'}$, from which we can find, as above,

$$n' = \frac{d(\log y)}{d(\log z)} = \frac{\frac{dy}{y}}{\frac{dz}{z}}.$$

Now, as m is the ratio of $d(\log y)$ to $\frac{dy}{y}$, it is also the ratio of $d(\log z)$ to $\frac{dz}{z}$; and

$$d(\log z) = \frac{mdz}{z}.$$

Thus we see that in any case in the same system the same ratio exists between the differential of the logarithm of a number and the differential of the number divided by the number. Therefore, m is a constant factor.

Again, $\frac{dy}{dx} = m\frac{1}{x}$ indicates the relative rate of change of a logarithm and its number. Now it is evident that the larger the base the slower the logarithm will change with reference to the number. But the factor $\frac{1}{x}$ varies inversely in the *number;* hence, m must vary with or be a function of the base.

491. *To produce the logarithmic series.*

The logarithmic series, which is the foundation of the usual method of computing logarithms, and of much of the theory of logarithms, is the development of $\log(1 + x)$. To develop $\log(1 + x)$, assume

$$\log(1+x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}, \quad (1)$$

in which x is a variable, and A, B, C, etc., are constants.

Differentiating (1), we have

$$\frac{mdx}{1+x} = Bdx + 2Cxdx + 3Dx^2dx + 4Ex^3dx + 5Fx^4dx + \text{etc.}$$

Dividing by dx,

$$\frac{m}{1+x} = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \text{etc.} \quad (2)$$

Differentiating (2), and dividing by dx, we have

$$-m\frac{1}{(1+x)^2} = 2C + 2\cdot 3Dx + 3\cdot 4Ex^2 + 4\cdot 5Fx^3 + \text{etc.} \quad (3)$$

Differentiating (3), and dividing by 2 and by dx, we have

$$m\frac{1}{(1+x)^3} = 3D + 3\cdot 4Ex + 2\cdot 3\cdot 5Fx^2 + \text{etc.} \quad (4)$$

Differentiating (4), and dividing by 3 and dx, we have

$$-m\frac{1}{(1+x)^4} = 4E + 4\cdot 5Fx + \text{etc.} \quad (5)$$

Differentiating (5), and dividing by 4 and dx, we have

$$m\frac{1}{(1+x)^5} = 5F + \text{etc.} \quad (6)$$

We have now gone far enough to enable us to determine the co-efficients A, B, C, D, E, and F, and these will probably reveal the law of the series.

As the equations are to be true for all values of x, and as the co-efficients A, B, C, etc., are constant, *i. e.*, have the same values for one value of x as for another, if we can determine their values for one value of x, these will be their values in all cases. Now, making $x = 0$, we have, from (1),

$$A = \log 1 = 0;$$

from (2), $B = m$;

from (3), $C = -\frac{1}{2}m$;

from (4), $D = \frac{1}{3}m$;

from (5), $E = -\frac{1}{4}m$;

from (6), $F = \frac{1}{5}m$.

These values substituted in (1) give

$$\log (1 + x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right),$$

the law of which is evident. This is the **logarithmic series**, and should be fixed in memory.

492. The Napierian system of logarithms is characterized by the modulus being 1 ($m = 1$). Hence, the **Napierian logarithmic series** is

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}$$

493. *The logarithms of the same number in different systems are to each other as the moduli of those systems.*

This is evident from the general logarithmic series. Thus,

the logarithm of $1 + x$, in a system whose modulus is m, is expressed,

$$\log_m (1 + x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right);$$

and the logarithm of the same number in a system whose modulus is m', is expressed,

$$\log_{m'} (1 + x) = m'\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right).$$

Now, as the number $(1 + x)$ is, by hypothesis, the same in both cases, x is the same. Hence, dividing one equation by the other, we have

$$\frac{\log_m (1 + x)}{\log_{m'}(1 + x)} = \frac{m}{m'}.$$

494. *The logarithm of a number in the Napierian system, multiplied by the modulus of any other system, gives the logarithm of the same number in the latter system.*

Or, the logarithm of a number in any system divided by the logarithm of the same number in the Napierian system, gives the modulus of the former system.

495. *To adapt the Napierian logarithmic series to numerical computation, so that it can be conveniently used for computing the logarithms of numbers.*

That $\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \text{etc.}$, is not in a practicable form for computing the logarithms of numbers will be evident if we make the attempt. Thus, suppose

we wish to compute the logarithm of 3. Making $x = 2$, we have

$$\log (1 + 2) = \log 3 = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \frac{2^5}{5} - \text{etc.},$$

a series in which the terms are growing larger and larger (a diverging series).

We wish a series in which the terms will grow smaller as we extend it (a converging series). Then the farther we extend the series, the more nearly shall we approximate the logarithm sought. To obtain such a series, substitute $-x$ for x in the Napierian logarithmic series, and we have

$$\log (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \text{etc.}$$

Subtracting this from the former series, we have

$$\log (1 + x) - \log (1 - x) = \log \left(\frac{1 + x}{1 - x}\right)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \text{etc.}\right)$$

Now, put $x = \dfrac{1}{2z + 1}$; whence,

$$1 + x = 1 + \frac{1}{2z + 1} = \frac{2z + 2}{2z + 1},$$

$$1 - x = \frac{2z}{2z + 1},$$

and

$$\frac{1 + x}{1 - x} = \frac{1 + z}{z}.$$

Hence, as $\log\left(\frac{1+z}{z}\right) = \log(1+z) - \log z$, substituting and transposing,

$$\log(1+z) = \log z + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \frac{1}{7(2z+1)^7} + \text{etc.}\right). \quad (A)$$

This series converges quite rapidly, especially for large values of z, and is convenient for use in computing logarithms.

496. ***To compute the Napierian logarithms of the natural numbers 1, 2, 3, 4, etc.***

1. It is necessary to compute the logarithms of *prime* numbers only, since the logarithm of a composite number is equal to the sum of the logarithms of its factors.

Therefore, beginning with 1, we know that

$$\log 1 = 0.$$

To compute the logarithm of 2, make $z = 1$, in series (A), and we have

$$\log(1+1) - \log 1 = \log 2$$
$$= 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \frac{1}{9\cdot 3^9} + \frac{1}{11\cdot 3^{11}} + \frac{1}{13\cdot 3^{13}} + \frac{1}{15\cdot 3^{15}} + \text{etc.}\right)$$

The numerical operations are conveniently performed as follows:

3	2.00000000		
9	.66666667	1	.66666667
9	.07407407	3	.02469136
9	.00823045	5	.00164609
9	.00091449	7	.00013064
9	.00010161	9	.00001129
9	.00001129	11	.00000103
9	.00000125	13	.00000009
	.00000014	15	.00000001

$$\therefore \log 2 = .69314718$$

2. To find log 3, make $z = 2$; whence,

$$\log 3 = \log 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} + \text{etc.}\right).$$

COMPUTATION.

5	2.00000000		
25	.40000000	1	.40000000
25	.01600000	3	.00533333
25	.00064000	5	.00012800
25	.00002560	7	.00000366
	.00000102	9	.00000011
			.40546510
		$\log 2 =$	.69314718
		$\therefore \log 3 =$	1.09861228

3. To find log 4.

$$\log 4 = 2 \log 2 = 2 \times .69314718 = 1.38629436.$$

4. To find log 5. Let $z = 4$; whence,

$$\log 5 = \log 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7} + \text{etc.}\right).$$

COMPUTATION.

9	2.00000000		
81	.22222222	1	.22222222
81	.00274348	3	.00091449
81	.00003387	5	.00000677
	.00000042	7	.00000006
			.22314354
		log 4 =	1.38629436
		∴ log 5 =	1.60943790

In like manner, we may proceed to compute the logarithms of the *prime* numbers from the formula, and obtain those of the composite numbers on the principle that the logarithm of the product equals the sum of the logarithms of the factors.

Thus, the Napierian logarithm of the base of the common system, 10, = log 5 + log 2 = 2.30258508.

497. *The modulus of the common system is*

.43429448 + .

Since the logarithm of a number, in any system, divided by the Napierian logarithm of the same number, is equal to the modulus of that system (**494**), we have

$$\frac{\text{Com. log } 10}{\text{Nap. log } 10} = \text{modulus of common system.}$$

But, com. log 10 = 1, and Nap. log 10 = 2.30258508, as found above. Hence,

$$\textit{Modulus of common system} = \frac{1}{2.30258508} = .43429448.$$

498. *The Napierian base is* 2.718281828.

Let e represent the base of the Napierian system. Then, by **492**),

$$\text{com. log } e : \text{Nap. log } e :: .43429448 : 1.$$

But the logarithm of the base of a system, taken in that system, is 1, since $a^1 = a$. Hence, Nap. log $e = 1$, and com. log $e = .43429448$. Now, finding from a table of common logarithms the number corresponding to the logarithm .43429448, we have

$$e = 2.718281828.$$

ANSWERS.

Art. 39.

1. 15.
2. 450.
3. 768.
4. 810.
5. 1000.
6. 240.
7. 1800.
8. 543.
9. 960.
10. 810.
11. 36.
12. 225.
13. 27.
14. 85.
15. 15.
16. 126.
17. 49.
18. 546.
19. 360.
20. 240.
21. 10.
22. 4.
23. 84.
24. 181.
25. 15.
26. 4.
27. 74.
28. 13.
29. 3.
30. $15\frac{1}{4}$.
31. 2.
32. $18\frac{1}{2}$.
33. 30.
34. 113.
35. 9.
36. 16.
37. 62.
38. 8000.

Art. 47.

1. 10; 15; 20.
2. 5; 10; 25,
3. 10.
4. 19.
5. 20; 10.
6. 4.
7. 6; 10.
8. 6; 7; 18; 7.
9. 7; 3; 8; 10.
11. 12 cts.
12. 70 cts.
13. $3.
14. 15 cts., bat; 50 cts., ball.
15. Jane, 17 qts.; Mary, 12 qts.
16. Father, 48 yrs.; son, 12 yrs.
17. Book, $1.05; slate, 30 cts.

18. 14 cts.

19. $180 each.

20. Paid, $5500, 1st ;
$2500, 2d ;
rec'd, $6300, 1st ;
$1600, 2d.

Art. 59.

1. $10a$.
2. $16x$.
3. $12b$.
4. $-5ax$.
5. $2abc$.
6. $5y$.
7. $10(a+b)$.
8. $8(b+c)$.
9. $(12+a)(x+y)$.
10. $(x+y+z)(a+b)$.
11. $x+3y$.
12. $2x-2y+3z$.
13. $9ab-6ax+15bd$.
14. $21abd-13abx-2cx$.
15. $b+a^2-xy+4$.
16. $3b^2+7a^3+5+5ab$.
17. $4ax^2-y+2b$.
18. $8c^2+bc+b-3$.
19. $3ab+a^2c+4$.
20. $6b^2-b-2c$.
21. $-a^2+9+6ac$.
22. y^3+b^2-6.
23. $12a+\frac{3}{4}c$.
24. $4x^2+y^2$.
25. $\frac{3}{5}m+7n+2y$.
26. $4a$.
27. $-2c+e$.
28. $\frac{33}{14}a+\frac{78}{65}ab$.
29. $4ab+3c+2b$.
30. $9ax-2ay$.
31. $28x^3+5x^2+18x+87$.
32. $1\frac{3}{4}a-\frac{1}{2}b$.
33. $5x-5y-2z$.
34. $3a^2bx+ab^2x+2abx^2-3a^2b^2x$.
35. $-\frac{5}{6}xy^2+3x^2y^2$.
36. $18a^3b^3c^3+5a^3b^3c^2$.
37. $8x^2+3y^2$.
38. $-x^2+x-1+1\frac{1}{4}xy$.

Art. 67.

1. $13a$.
2. $3ax$.
3. $8x$.
4. $6abc$.
5. $15b$.
6. $-3ax$.
7. $a-7b$.

8. $x - 4y + 2z$.
9. $2x + y + 4z$.
10. $4a - 4b + 4c$.
11. $3x^4 - x^3 - 14x + 18$.
12. $5a^3 - 5a^2b - 2bc + 15$.
13. $2b^2c + 4abx + 3$.
14. $13ax - 6by + dx$.
15. $6bcx + 5aby$.
16. $8bx - 6acy + bxy + bc$.
17. $18ab + 5eg$.
18. $-6y^2$.
19. $4xy$.
20. $-2y^2$.
21. $5a + 4c$.
22. $x - 3y - 2z$.
23. y.
24. $8ax + 4bx + cx - d$.
25. $6am - 2yz$.
26. $-\frac{1}{10}a$.
27. $.95x - .989y - 3x^2$.
28. $\frac{3}{15}x - \frac{1}{15}y + \frac{13}{28}z$.
29. $1\frac{1}{2}a + 4\frac{1}{3}b - 4\frac{9}{10}c$.
30. $18 + ax - 5by + 2c$.
31. $\frac{2}{3}a + \frac{1}{2}c$.
32. $-2z$.
33. $7ax + 9by - 2c$.
34. $3a - 5b - 2c$.

Art. 70.

1. 12.
2. 3.
3. 4.
4. 1.
5. 16.
6. 70.
7. a.
8. $27a - 14b$.
9. $2c$.
10. $a + 3b - 2c$.
11. $17a - 5c + 2x$.
12. $2x + 2y - 2z$.
13. $-15a + 8b - c$.
14. $8d + 2e - 4a - 13c$.
15. $40a + 6b - 5f + 5c - 3h$.
16. $a - b + c + d - e$.
17. $a - b + 2c - d$.
18. $-a - 2x$.
19. $2a - b$.
20. $7x - 2y - 6z$.
21. a.
22. 0.
23. $c + y^2$.
24. $-2b + 2ab - c$.

Art. 81.

1. $15m^2xy$.
2. $56a^3y^4$.
3. $24a^3b^5$.

4. $70a^2cx^4y^3$.
5. $-20a^3b^3$.
6. $-35a^3bx^3$.
7. $-10a^3b^5c^2$.
8. $21a^5x^3y$.
9. $36m^3n^3$.
10. $-56x^3y^3$.
11. $-91a^5c^5$.
12. $125a^4bm^3n^3$.
13. $120a^6b^3c^5x^2$.
14. $24a^3bx^4y^7z^2$.
15. $-90a^4b^3c^4m^5$.
16. $-90a^2x^4y^9$.
17. $6a^8x^5y^8$.
18. $-120a^7b^8c^8$.
19. $52a^4b^6c^5$.
20. $60a^5b^3c^3$.
21. $100a^3x^8$.
22. $100m^5n^3$.
23. $6a^2b^{\frac{7}{2}}c^3$.
24. $310a^3y^2z^2$.
25. $-45a^2m^3n^3y$.
26. $-30a^4x^3$.
27. $24m^{13}n^{23}x^3$.
28. $-24a^3c^7y^5$.
29. $120a^5c^6y^7$.
30. $48a^4b^4c^7$.

Art. 83.

1. $a^2 + 2ab + b^2$.
2. $6x^2 + 13xy + 6y^2$.
3. $6a^2b^2 - ab^3 - 12b^4$.
4. $2a^2 + ab - 4ac - b^2 + 2bc$.
5. $x^4 + x^2z - y^4 - y^2z$.
6. $a^4 - a^2b^2 + 2ab^3 - b^4$.
7. $a^4 + 3a^2b^2 + 4b^4$.
8. $a^5 - 41a - 120$.
9. $6x^4 - 96$.
10. $81x^4 - y^4$.
11. $a^5 + 32b^5$.
12. $x^6 + x^5 - 11x + 9$.
13. $a^9 - 6a^4 + 25a^3 - 4a^2 + 32a + 4$.
14. $m^6 - 6m + 5$.
15. $x^6 - 2x^3 + 1$.
16. $2a^7 - 7a^6b + 7a^5b^2 + a^4b^3 - 15a^3b^4$.
17. $a^6 - a^4b^2 + a^2b^4 - b^6$.
18. $a^3 + b^3 + c^3 - 3abc$.
19. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$.
20. $189a^4 - 91a^3b + 62a^2b^2 - 13ab^3 + 5b^4$.
21. $x^5 - 32y^5$.
22. $x^9 - 15x^6 + 56x^5 + 56x^4 - 15x^3 + 1$.

23. $3x^5 - 46x^3y^2 + 43x^2y^3 - 36y^5$.

24. $m^9 - 20m^4 - 27m^3 + 25m^2 - 135m + 21$.

25. $84a^6 - 437a^4b^2 + b^6$.

26. $243x^5 - y^5$.

27. $x^9 + 131x^4 + 120x^2 - 81x + 21$.

28. $x^5 - 125x^3 + 3004x + 5040$.

29. $x^6 - 98x^4 + 2401x^2 - 14400$.

30. $a^6 - b^6$.

31. $x^4 - 25x^2 + 144$.

32. $a^6 - a^5b + a^2b^4 - b^6$.

33. $1 - x^8$.

Art. 107.

1. ad.

2. $2a^4b^2$.

3. $7b$.

4. $2a^3bcd$.

5. $2axy$.

6. $-2abc$.

7. $3y^3 + 6x^3y^7$.

8. $2ab + 4a^3b - c$.

9. $4a^3c + 3d^3 - 2ae$.

10. $3a^4c - a + 6a^2c$.

11. $3a + 4b - 5c$.

12. $6m - 7n$.

13. $3ab - 4c$.

14. $a + 4a^2bn - 7n^2$.

15. $x - 4yz^2 + 5az^4$.

16. $b + 1 - abc + a^2b^3c^2$.

17. $-3ax^2 + 5bxy - 7cy^2$.

18. $mx^3y - nxy^3$.

19. $3ab^3 - 5c + 3\frac{1}{2}a^2bx$.

20. $-x^2y^2 + 8a^2xy - 5a^4$.

21. $3a^3bc^2 - 2a^4b^2c^3 - a^5b^3c^4$.

22. $3xy^2 - x^3y^5 - 3y^2 - 5xy^3$.

23. $3a^2x^2 - ax + 2 - \frac{9}{ax}$.

24. $6a^2 - 1\frac{1}{4}a + 4 + \frac{5}{a}$.

Art. 109.

1. $x + y$.

2. $x - y$.

3. $x^2 - 2xy + y^2$.

4. $x^2 + 3x + 10$.

5. $x^3 + 3x^2 - 4$.

6. $x^2 + x + 3$.

7. $x^2 - 3x - y$.

8. $2a - 5b$.
9. $x^2 + 2x + 1$.
10. $x + 2$.
11. $x - 2$.
12. $a^2 - 4b^2$.
13. $x^2 + 10x + 14$.
14. $x^2 - 9x - 10$.
15. $x^2 - 3x + 7$.
16. $x^3 - x^2y + xy^2$.
17. $x^2 - 2x + 2$.
18. $x^2 - 4x + 8$.
19. $9x^2 - 6xy + 4y^2$.
20. $x^3 + 2x^2 + 4x + 8$.
21. $x^4 + 2x^3 + 3x^2 + 2x + 1$.
22. $7a^4 + 21a^3b + 8a^2b^2 + 3ab^3 + b^4$.
23. $a^2 + ab - 2b^2$.
24. $2x^3 + 4x^2 + 8x + 16$.
25. $x^4 + 2x^3 + x^2 - 4x - 11$.
26. $x^3 + 3x^2y + 9xy^2 + 27y^3$.
27. $x^2 - 3xy - 2y^2$.
28. $2x - 4y$.
29. $x^4 + 2x^3 + 3x^2 + 4x + 5$.
30. $1 - 3x + 2x^2 - x^3$.
31. $ab - a - b + 1$.
32. $ab - ac + bc$.
33. $4a^2 - 6ab - 8b^2$.
34. $3a + 2b + c$.
35. $x + y + z$.
36. $a^3 + 2a^2b + 2ab^2 + b^3$.
37. $1 - 2x + x^2$.
38. $1 + 3y + 9y^2 + 27y^3$
39. $a^3 - 4a^2 + 9a - 8$.
40. $x^4 + 2x^3 + 3x^2 + 4x + 5$.
41. $x^4 - 3x^2 + 4x + 1$.
42. $x^4 + 2x^3 + 3x^2 + 2x + 1$.
43. $x^4 - 3x^3 - 2x^2 - 4$.
44. $2x^3 - 6x^2 + 3x - 1$.
45. $3x^3 - 15x^2 + 20x - 12$.
46. $a^6 - 2a^5 + 4a^4 - 5a^3 + 5a^2 + 5$.
47. $x^6 - 2x^5 + 4x^4 - 3x^3 + 7x^2 - 16x + 32$.
48. $x^4 - 3x^3y + 10x^2y^2 - 34xy^3 + 115y^4$.
49. $8x^5 - 8x^4 + 4x^3 - 2x^2 + 2x - 1$.
50. $m^6 - 2m^5 + 4m^4 - 5m^3 + 6m^2 - 4m + 1$.
51. $x^5 + 3x^4y + 13x^3y^2 - 5x^2y^3 + 22xy^4 - 19y^5$.

52. $a^5 - 2a^4 - 13a^3 - 6a^2 + a - 5$.

53. $m^3 - 7m^2n - 4mn^2 + 3n^3$.

Art. 129.

1. $(a^2 + ay + y^2)(a^2 - ay + y^2)$,
$(m^2 + mn + n^2)(m^2 - mn + n^2)$,
$(p^2 + pq + q^2)(p^2 - pq + q^2)$.

2. $(x^2 + 3x + 9)(x^2 - 3x + 9)$,
$(4a^2 + 2ax + x^2)(4a^2 - 2ax + x^2)$,
$(x^4 + x^2y^2 + y^4)(x^4 - x^2y^2 + y^4)$.

3. $(x^4 + x^2y + y^2)(x^4 - x^2y + y^2)$,
$(m^2 + mn^2 + n^4)(m^2 - mn^2 + n^4)$,
$(9a^2 + 6a + 4)(9a^2 - 6a + 4)$.

4. $(p^4 + p^2 + 1)(p^4 - p^2 + 1)$,
$(m^2 + m + 1)(m^2 - m + 1)$,
$(x^2 + 4x + 16)(x^2 - 4x + 16)$.

Art. 131.

1. $(x + 2)(x + 1)$, $(x + 3)(x + 7)$, $(a + 2)(a + 6)$.

2. $(x + 7)(x + 8)$, $(x + 1)(x + 20)$,
$(m + 6)(m + 7)$.

3. $(x + 4)(x + 10)$, $(x + 4)(x + 7)$,
$(m + 5)(m + 12)$.

4. $(a - 4)(a - 5)$, $(a - 3)(a - 8)$,
$(m - 3)(m - 7)$.

5. $(x - 1)(x - 8)$, $(x - 3)(x - 12)$,
$(x - 8)(x - 12)$.

6. $(x - 7)(x - 12)$, $(x - 4)(x - 11)$,
$(x - 6)(x + 17)$.

7. $(y + 3)(y - 5)$, $(y + 7)(y - 11)$,
$(x + 5)(x - 12)$.

8. $(x - 3)(x + 14)$, $(x - 4)(x + 15)$,
$(x - 10)(x + 21)$.

9. $(x^2 + 2)(x^2 - 11)$, $(a^2 - 3)(a^2 + 8)$,
$(x^2 + 5)(x^2 - 13)$.

10. $(a + 2)(a + 25)$, $(y + 5)(y - 10)$,
$(x - 2)(x + 45)$.

11. $(a + 3)(a + 5)$, $(c^2 + 23)(c^2 - 3)$,
$(b^2 - 12)(b^2 + 10)$.

Art. 133.

1. $(5x + 1)(x + 2)$, $(5x + 2)(x + 1)$,
$(2y + 5)(y + 3)$.

2. $(a + 7)(3a + 1)$, $(x + 7)(5x + 1)$,
$(m + 3)(11m + 1)$.

3. $(x - 7)(2x - 3)$, $(n - 5)(2n - 7)$,
$(n + 3)(2n - 7)$.

4. $(5m - 1)(3m - 11)$, $(3x - 1)(13x - 7)$,
$(y + 2)(3y - 7)$.

5. $(15b - 1)(b + 3)$, $(7y - 13)(y - 3)$,
$(3x - 5)(5x + 6)$.

6. $(x + 6)(2x - 11)$, $(3y - 13)(y + 5)$,
$(7a - 1)(3a + 2)$.

7. $(13m - 7)(7m + 3)$, $(5a + 3)(3a - 2)$,
$(5x + 1)(x - 1)$.

8. $(2x^2 - 5)(x^2 + 2)$, $(5n^2 - 7)(3n^2 + 4)$,
$(11x^3 + 7)(2x^3 - 5)$.

9. $(4m^2 - 9)(m^2 - 4)$, $(8x^3 - 27)(x^3 + 1)$,
$(2x^2 - 3)(3x^2 + 2)$.

Art. 135.

1. $(m + 2)(x + y)$, $(x + 1)(a + b)$.
2. $(2x + 3)(2 + a)$, $(a + b)(c + d)$.
3. $(a - 3)(b + 1)$, $(y - 2)(3x - 1)$.
4. $(2x + 3y)(x - m)$, $(2x + m)(x - 3)$.
5. $(mq - x + p)(q - x)$.
6. $(p + q - s)(a - c)$.
7. $(r - c + 2n)(a + 2b)$.
8. $(a - c + 2b)(x + y)$.
9. $(a + 2c - 4ac)(3b + d)$.
10. $(2a - b - c)(m - d)$.
11. $(4c + 3n + 2r)(a - 2b)$.
12. $(n + 2r - cd)(a + bs)$.
13. $(2x - y - z)(a + c)$.
14. $(2x + 3y)(3n - p + 2s)$.
15. $(2x - 2y)(3a + 2b - 1)$.
16. $(3x - 2y - 5)(7 + n)$.

Art. 137.

1. $(x + 2)(x + 3)(x + 1)$.
2. $(x - 1)(x - 2)(x + 8)$.
3. $(x - 2)(x + 3)(x + 4)$.
4. $(x - 3)(x + 5)(x + 10)$.
5. $(x - 1)(x - 5)(x + 7)$.
6. $(x + 2)(x - 3)(x + 5)$.
7. $(x + 1)(x + 3)(x - 7)$.
8. $(x - 2)(x - 3)(x - 5)$.

9. $(x - 1)(x + 1)(x + 8)$.
10. $(x - 6)(x + 4)(x + 2)$.
11. $(x + 6)(x - 3)(x - 3)$.
12. $(x + 10)(x - 5)(x - 5)$.
13. $(x - 7)(x + 5)(x + 2)$.
14. $(x + 3)(x + 5)(x - 25)$.
15. $(x + 3)(x + 5)(x + 11)$.
16. $(x - 1)(x + 9)(x - 10)$.

Art. 139.

1. $(x + a + 6)(x + a)$.
2. $(x - 2a + 9)(x - 2a)$.
3. $(m - 5 - n)(m - 5)$.
4. $(x - 7 - 5a)(x - 7)$.
5. $(a - 3x + 2)(a - 3x)$.
6. $(y + z + by)(y + z)$.
7. $(b^2 - 2bc + c^2 - 4a)(b - c)$.
8. $(5n - m - 6r)(5n - m)$.
9. $(3x - 4y - 5z)(3x - 4y)$.
10. $(7x - 2 + 3y)(7x - 2)$.
11. $(m^2 - 3n^2 - 6p)(m^2 - 3n^2)$.
12. $(2n + 1 - 5m)(2n + 1)$.
13. $(x + 3y - 2c)(x + 3y)$.
14. $(x - 7y - 12)(x - 7y)$.
15. $(3x - 2y - 4)(3x - 2y)$.
16. $(ax - 3b + 11c)(ax - 3b)$.
17. $(a^2 - 2b - 5c)(a^2 - 2b)$.

18. $(2a - b - 6c)(2a - b)$.

19. $(p + 2q - 12r)(p + 2q)$.

20. $(2c - 3d - 3ab)(2c - 3d)$.

21. $(7a + 3b - 13)(7a + 3b)$.

22. $(5a - 2b - 2ab)(5a - 2b)$.

23. $(am + 3 + 5x)(am + 3)$.

24. $(3x + 2 + 6a)(3x + 2)$.

25. $(5x + 3y - 4)(5x + 3y)$.

Art. 140.

1. $(a^2 + 9)(a + 3)(a - 3)$.

2. $(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.

3. $(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$.

4. $(m - n)(m^4 + m^3n + m^2n^2 + mn^3 + n^4)$.

5. $(x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$.

6. $(a - 5)(a^2 + 5a + 25)$.

7. $(a - 1)(a^4 + a^3 + a^2 + a + 1)$.

8. $(x + 4)(x^2 - 4x + 16)$.

9. $(a^6 + b^6)(a + b)(a^2 - ab + b^2)(a - b)(a^3 + ab + b^2)$.

10. $(x + 2)(x^4 - 2x^3 + 4x^2 - 8x + 16)$.

11. $(x + 2n)(x^2 - 2nx + 4n^2)$.

12. $(a + 6)(a^2 - 6a + 36)$.

13. $(x + 10)(x^2 - 10x + 100)$.

14. $(x^2 + 10)(x^2 - 10)$.

15. $(1 + x)(1 - x + x^2 - x^3 + x^4 - x^5 + x^6)$.

16. $(x - y)(x^{14} + x^{13}y + x^{12}y^2 + x^{11}y^3 + x^{10}y^4 + x^9y^5 + x^8y^6 + x^7y^7 + x^6y^8 + x^5y^9 + x^4y^{10} + x^3y^{11} + x^2y^{12} + xy^{13} + y^{14})$.

17. $(x + 7)(x^2 - 7x + 49)$.

18. $(a + 1)(a^4 - a^3 + a^2 - a + 1)$.

19. $(x^2 + y^2)(x + y)(x - y)$.

20. $(a^6 + 1)(a + 1)(a^2 - a + 1)(a - 1)(a^2 + a + 1)$.

21. $(ax + by)(ax - by)$.

22. $(mn + pq)(mn - pq)$.

23. $(x^2 + 25)(x + 5)(x - 5)$.

24. $(b + 2)(b^2 - 2b + 4)(b - 2)(b^2 + 2b + 4)$.

25. $(x + 3)(x^4 - 3x^3 + 9x^2 - 27x + 81)$.

26. $(x + 2y)(x + 2y)$.

27. $(a - 3b)(a - 3b)$.

28. $(2a - 3x)(2a - 3x)$.

29. $(4m - 5)(4m - 5)$.

30. $(x - 1)(x - 9)$.

31. $(x - 9)(x - 25)$.

32. $(x + 7)(x - 2)$.

33. $(x + 5)(x - 5)(x + 1)(x - 1)$.

34. $(a + 4)(a + 2)$.

35. $(m + 5)(m - 5)(m + 2)(m - 2)$.

36. $(x - 7)(x + 2)$.

37. $(2a + 7)(8a - 3)$.

38. $(3b - 8)(b + 2)$.

39. $(3y - 4)(y - 7)$.

40. $(2z - 3)(z - 2)$.

41. $(a + 3)(5a + 7)$.

42. $(8x^2 - 3)(2x^2 - 7)$.

43. $(2p - 3)(5p - 13)$.

44. $(x - 8)(x - 8)$.

45. $(5m - 3)(2m + 5)$.

46. $(3x + 2)(2x - 1)$.

47. $(2x^3 - 3)(x^3 + 2)$.

48. $(x^2 + 4)(4x^2 - 3)$.

49. $(2n - 5)(2n - 5)$.

50. $(7xy + 11)(7xy + 11)$.

51. $(a - b + c)(a + b - c)$.

52. $(b + x)(a + x)(a - x)$.

53. $(x - y - 2z)(x - y + 2z)$.

54. $(3x + 2y - 5z)(3x - 2y + 5z)$.

55. $(a - x)(x + b)(x - b)$.

56. $(x^2 - yz - 4z)(x^2 - yz + 4z)$.

57. $(a^2 - b^2 - c^2 + d^2)(a^2 - b^2 + c^2 - d^2)$.

58. $(x + y - z)(a - c)$.

59. $(3x - 1 - 5y)(3x - 1)$.

60. $(m - n + 2b)(x + y)$.

61. $(2x - m + 3n)(3y + 2x)$.

62. $(x - 4)(x + 5)(x + 1)$.

63. $(2y + 7 - 3z)(2y + 7)$.

64. $(x - 2)(x - 3)(x - 5)$.

65. $(x - 3)(x - 10)(x + 4)$.

66. $(b - c - a + y)(b - c + a - y)$.

67. $(x - 3)(x + 4)(x - 5)$.

68. $(4x - 5y - 6)(4x - 5y)$.

69. $(x + 4)(x + 7)(x - 5)$.

70. $(a^2 - bc)(a + b + c)$.

71. $(a + x)(b - x)(b + y)$.

72. $(2m + 3n - c)(x - 2r)$.

73. $(a - d - b + c)(a - d + b - c)$.

74. $(a + bc + 5m)(a + bc)$.

75. $(a + 3m)(a + 2m)(a - m)$.

76. $(x - 5y)(x + 4y)(x + 3y)$.

77. $(a + 2b + c)(x - y)$.

Art. 144.

1. $3ax$.
2. $abuv$.
3. $4ab$.
4. $9x^2y^2z$.
5. $a + b$.
6. $x + 1$.
7. $m + n$.
8. $a + 3$.
9. $x - 5$.
10. $x - 7$.
11. $m + 6$.
12. $b - 11$.
13. $5x + 25$.
14. $x - 9$.
15. $x - 11$.
16. $2x + 5$.
17. $3x - 2$.
18. $x + 3$.
19. $x + 2$.
20. $x + 3$.
21. $x - 5$.
22. $2y - 6$.
23. $a - 3b$.
24. $a - 2$.

Art. 147.

1. $x + 3$.
2. $x - 1$.
3. $x - 1$.
4. $x - 2$.
5. $2x + 5$.
6. $x^3 - 4x + 3$.
7. $5x^3 - 1$.
8. $x^2 - 2x + 1$.
9. $x^4 - 7x^2 + 1$.
10. $x^4 + x^3 - x - 1$.
11. $x^2 - 2x + 1$.
12. $x^3 - 3x^2 + 5x - 2$.
13. $x^3 - 5x^2 + 5x - 3$.
14. $x^4 - 4x^3 + 2x - 3$.
15. $x^3 + 4x^2 - 2x + 3$.
16. $a^2 + 4ac + 7c^2$.
17. $x^3 + 3x^2 + 3x + 1$.
18. $x^2 + 5x + 3$.
19. $x^3 - 4x^2 + 4x - 1$.
20. $x^2 + 4x + 7$.
21. $x^2 + 4x + 7$.
22. $x + 3$.

Art. 151.

1. $12a^4b^3$.
2. $45a^4b^7$.
3. $105x^3y^3z^2$.
4. $42x^3y^2z^2$.
5. $2(4x^2 - 1)$.
6. $x^4 + 2x^3 - 8x - 16$.
7. $48xy(x^3 - y^3)$.
8. $x^3 - 4x^2 - 47x + 210$.
9. $x^3 - 13x - 12$.
10. $x^3 - 10x^2 - 123x + 1188$.
11. $x^3 - 6x^2 - 19x + 84$.
12. $x^5 - 6x^3 + 6x^2 - 7x + 6$.
13. $24x^3 - 14x^2 - x + 1$.

14. $12x^4 - 20x^3 + 5x^2 + 5x - 2$.

15. $a^4 - a^3 + 6a^2 + a - 7$.

16. $105ab^2(a^2 - b^2)$.

17. $x^8 - 1$.

18. $x^4 + 5x^3 + 5x^2 - 5x - 6$.

19. $x^4 + 3x^3 - 7x^2 - 27x - 18$.

20. $x^5 - 8x^4 + 50x^2 - x - 42$.

21. $x^4 - 10x^3 + 35x^2 - 50x + 24$.

Art. 158.

1. $\dfrac{3ay + x}{y}$.

2. $\dfrac{2x + a}{x}$.

3. $\dfrac{4ab - a}{b}$.

4. $\dfrac{2a^2x^2 + a^3}{x}$.

5. $\dfrac{1 + 2x}{1 + x}$.

6. $\dfrac{2a^2 - 2ab + 4b^2}{a - b}$.

7. $\dfrac{2a^2}{a - x}$.

8. $\dfrac{3x^2}{x^2 - 1}$.

9. $\dfrac{a^2x + 4a}{a + x}$.

10. $\dfrac{5bx - 2a^2x}{a^2}$.

11. $\dfrac{b}{a + c}$.

12. $\dfrac{a^2 - ab + 3a}{a + 1}$.

13. $\dfrac{x^6 - x^5y + 2y}{x - y}$.

14. $\dfrac{a^2 - b^2}{a - 3b}$.

15. $\dfrac{am^2}{m - n}$.

16. $\dfrac{2a^2 - 5}{a + x}$.

17. $\dfrac{x^2y^2}{x^2 + xy + y^2}$.

18. $\dfrac{x^3 - 5}{x + 5}$.

19. $\dfrac{x^4 + 5}{x^2 + 2x - 1}$.

20. $\dfrac{x^2 + x - 3}{x^2 + x + 1}$.

21. $\dfrac{6a^6x^3y^2}{2a^2x^2}$.

22. $\dfrac{a^4 - x^4}{a^2 + x^2}$.

23. $\dfrac{20ax^3y^3z^2}{4axz}$.

24. $\frac{ax(x^2-y^2)}{x-y}$.

25. $\frac{a^4-b^4}{a-b}$.

Art. 160.

1. $2a+\frac{5a}{9}$.

2. $3abc+\frac{3abc}{4}$.

3. $4ac+\frac{4c}{9}$.

4. $2a+\frac{3b}{4a}$.

5. $2x-\frac{5y}{6x}$.

6. $2ax-\frac{x^3}{a}$.

7. $x+\frac{2}{x+3}$.

8. $2x-\frac{1}{x-3}$.

9. $x-2y$.

10. $x^2+xy+y^2+\frac{2y^3}{x-y}$.

11. $a^2-b^2+\frac{2b^4}{a^2+b^2}$.

12. $a^2-ab+b^2-\frac{2b^3}{a+b}$.

13. $x-7+\frac{12}{x+7}$.

14. x^3-x^2+x-1.

15. a^2+b^2.

16. $x^4+2+\frac{3}{x-1}$.

17. $x^2+4x+7+\frac{28x^2+x-2}{x^3-4x^2+4x-7}$.

18. $2x^2+5x+3+\frac{x^2}{x^3-5x^2+5x-3}$.

Art. 164.

1. $\frac{1}{3xy^2}$.

2. $\frac{2yz}{7x}$.

3. $\frac{3x^2}{4y^2z}$.

4. $\frac{5mnxz}{7pqy}$.

5. $\frac{2ab}{9cde}$.

6. $\frac{2x}{3a}$.

7. $\frac{4y^2z}{5xu^2}$.

8. $\frac{bc}{5ax}$.

9. $\frac{cy^4}{5abx^3}$.

10. $\frac{a+b}{2b}$.

11. $\frac{a+b}{a-b}$.

12. $\frac{ax}{3ax-2y^2}$.

13. $\frac{x}{a-b}$.

14. $\frac{3a}{a-x}$

15. $\frac{a+b}{a-b}$.

16. $\frac{a^2-ab+b^2}{a-b}$.

17. $\frac{x}{x^2+y^2}$.

18. $\frac{x+2}{x+5}$.

19. $\frac{2ax}{ax-3y^2}$.

20. $\frac{2(x+y)}{3(x-y)}$.

21. $\frac{x+3}{x-7}$.

22. $\frac{x+7}{x-5}$.

23. $\frac{x-1}{x+1}$.

24. $\frac{x^2}{x^2+y^2}$.

25. $\frac{x^3}{x^2-1}$.

26. $\frac{x-2}{x+3}$.

27. $\frac{x+3}{x^2-2x+5}$.

28. $\frac{3x-4}{4x-3}$.

29. $\frac{1}{17(a-3b)}$.

30. $\frac{x^2}{x-a}$.

31. $\frac{a}{bx+cy}$.

32. $\frac{3m}{n-r}$.

33. $\frac{1}{1-x}$.

34. $\frac{2x-3}{2x-7}$.

35. $\frac{2m-3x}{5m-3x}$.

36. $\frac{3x+2}{3x-2}$.

37. $\frac{2x+1}{x-2}$.

38. $\frac{x+1}{x-4}$.

39. $\frac{x+7}{x+8}$.

40. $\frac{3x-5}{5x+7}$.

41. $\frac{2x+5}{5x+1}$.

42. $\frac{x^2-y^2}{x^6+y}$.

43. $\frac{1}{x-1}$.

Art. 166.

1. $\frac{9x^2}{12x^3}$, $\frac{8x}{12x^3}$, $\frac{5}{12x^3}$

2. $\frac{x^2z}{xyz}$, $\frac{xy^2}{xyz}$, $\frac{yz^2}{xyz}$.

3. $\frac{c}{abc}, \frac{2b}{abc}, \frac{3a}{abc}$.

4. $\frac{adf}{bdf}, \frac{bcf}{bdf}, \frac{bde}{bdf}$.

5. $\frac{6ay^2}{8x^2y^2}, \frac{6ax^2}{8x^2y^2}, \frac{40b}{8x^2y^2}$

6. $\frac{(a-x)^2}{a^2-x^2}, \frac{(a+x)^2}{a^2-x^2}$

7. $\frac{(a-b)^2}{a^2-b^2}$,

$\frac{(a+2b)(a+b)}{a^2-b^2}$,

$\frac{a^2}{a^2-b^2}$.

8. $\frac{(a-b)^2}{a^3-b^3}, \frac{a^2+b^2}{a^3-b^3}$,

$\frac{b(a^2+ab+b^2)}{a^3-b^3}$.

9. $\frac{12(1-x)}{4(1-x^2)}, \frac{5(1-x)}{4(1-x^2)}$.

$\frac{8x}{4(1-x^2)}$.

10. $\frac{x(x^3+x^2y+xy^2+y^3)}{x^4-y^4}$.

$\frac{y(x^3-x^2y+xy^2-y^3)}{x^4-y^4}$.

$\frac{xy(x^2+y^2)}{x^4-y^4}$,

$\frac{y^2(x^2-y^2)}{x^4-y^4}$.

11. $\frac{a(a^2+ab+b^2)}{a^4+a^2b^2+b^4}$,

$\frac{b(a^2-ab+b^2)}{a^4+a^2b^2+b^4}$,

$\frac{c}{a^4+a^2b^2+b^4}$.

12. $\frac{(x^2-1)(x+3)}{(x-5)(x-3)(x+3)(x+1)}$,

$\frac{(x+3)(x^2-9)}{(x-5)(x-3)(x+3)(x+1)}$,

$\frac{(x-5)^2(x-3)}{(x-5)(x-3)(x+3)(x+1)}$.

13. $\frac{(x^2-9)(x+3)}{(x^2-4)(x+3)^2}, \frac{(x^2-4)(x+3)}{(x^2-4)(x+3)^2}, \frac{x^4-16}{(x^2-4)(x+3)^2}$.

14. $\dfrac{5(x^2+x-6)}{(x^2+x-20)(x^2+x-6)}$,
$\dfrac{3(x^2+3x-10)}{(x^2+x-20)(x^2+x-6)}$,
$\dfrac{2(x^2+x-20)}{(x^2+x-20)(x^2+x-6)}$.

15. $\dfrac{b^2-x^2}{(a-x)(a-y)(b-x)}$, $\dfrac{a^2-x^2}{(a-x)(a-y)(b-x)}$.

Art. 172.

1. $\dfrac{ax+bx+c}{x^2}$.

2. $\dfrac{5a+b}{18}$.

3. $\dfrac{7x^2+y^2}{36}$.

4. $\dfrac{15a+by+9}{3ay}$.

5. $\dfrac{3x-10a}{4ax}$.

6. $\dfrac{a+b}{12}$.

7. $\dfrac{2a}{a^2-b^2}$.

8. $\dfrac{12x-2y}{x^2-y^2}$.

9. $\dfrac{an^2x-pc}{b^2cn^2}$.

10. $\dfrac{an+b}{m^2-n^2}$.

11. $\dfrac{4xy}{x^2-y^2}$.

12. $\dfrac{2(ax-ny)}{x^2-y^2}$.

13. $\dfrac{a+b+c}{abc}$.

14. $\dfrac{1}{m^2-1}$.

15. $\dfrac{6-x^3}{x^2-9}$.

16. $\dfrac{5}{x-5}$.

17. $\dfrac{12x}{1-9x^2}$.

18. $\dfrac{abcx^3+cx+2b}{bcx^2}$.

19. $\dfrac{a^2(15m+10ambx^2+6a^2b^2x^2y)}{120mx^3}$.

20. $\frac{x^3+y^3}{x-y}$.

21. $\frac{x+y}{x-y}$.

22. $\frac{2a^2}{x(x^2-a^2)}$.

23. $\frac{4a}{a+x}$.

24. $\frac{2mx+n}{x^2-y^2}$.

25. $\frac{2x^3-2x^2-18x+8}{x^3+3x^2-4x-12}$.

26. $\frac{2a^2+9c^2}{6ac}$.

27. 2.

28. $\frac{2}{m+n}$.

29. $\frac{2ab^2}{a^4-b^4}$.

30. $\frac{2}{x(1-4x^2)}$.

31. $\frac{b(a+b)}{x^2-b^2}$.

32. $\frac{2x-3}{x(4x^2-1)}$.

33. $\frac{2}{(x+1)(x+2)(x+3)}$.

34. $\frac{4(x+10)}{x^4-16}$.

35. $\frac{2x^2-9x+44}{x^3+64}$.

36. $\frac{1}{a+b}$.

37. $\frac{2a^2b^2}{a^3-b^3}$.

38. $\frac{1}{x-7}$.

39. $\frac{1}{x^2-y^2}$.

40. $\frac{9}{(x-1)(x+2)^2}$.

41. $\frac{4ab}{(a-b)^2}$.

42. $\frac{4x}{x+y}$.

43. $\frac{4a^2}{b^2-a^2}$.

44. 1.

45. 0.

46. $\frac{c}{(x-a)(x-b)}$.

47. $\frac{x^2+x+1}{x^3(x^2+1)^2}$.

48. $\frac{2x-3}{(x^2-1)(2x+3)}$.

49. 0.

50. -3.

51. 0.

52. $\frac{24b^4}{a(a^2-b^2)(a^2-4b^2)}$.

Art. 175.

1. $\frac{a^2mn}{bc}$.

2. $\frac{4c}{5a}$.

3. $\frac{5n}{4bcx}$.

4. 1.

5. 8.

6. $\frac{4a^2+4ab}{7}$.

7. $6x$.

8. $\frac{4a(a+b)}{7}$.

9. $\frac{9x^2}{2}$.

10. $\frac{3x-1}{x-2}$.

11. $x-a$.

12. $\frac{a^4-b^4}{ab}$.

13. $\frac{(a-b)b}{(a+b)x}$.

14. $\frac{a^4-x^4}{a^2x}$.

15. $\frac{b}{a^2-ab+b^2}$.

16. $\frac{x^3-y^3}{y(x^2+y^2)}$.

17. $\frac{1}{(x-1)(x+2)}$.

18. $\frac{x+3}{y-2}$.

19. $\frac{1-y}{x}$.

20. $\frac{3x}{4y}$.

21. 1.

22. $\frac{a-4}{x+6}$.

23. $a+b$.

24. $\frac{ax}{a^2-x^2}$.

25. $\frac{a^2+b^2}{a}$.

26. $\frac{a-x}{m+n}$.

27. $\frac{x+2}{x-2}$.

28. $\frac{x^4}{a^4}+\frac{x^2}{a^2}+1$.

29. $\frac{x^2-11x+28}{x^2}$.

30. $\frac{4ab}{a^2-b^2}$.

31. $\frac{a^2(a-b)}{x}$.

32. $\frac{x^2}{a^2 - x^2}$.

33. $\frac{1 - y}{x}$.

34. $-ax$.

35. $\frac{4}{3}$.

36. $\frac{x^2 - y^2}{2(x^2 + y^2)}$.

37. $\frac{1}{(a - b)^2}$.

Art. 177.

1. $\frac{ad}{bc}$.

2. $\frac{ay}{bx}$.

3. $\frac{a^2}{m^2nq}$.

4. $\frac{2(x + y)}{3(x - y)}$.

5. $\frac{a^2}{b(a - b)}$.

6. $\frac{x^2 + xy + y^2}{x^2 + y^2}$.

7. $\frac{2y}{x^2 - xy + y^2}$.

8. $\frac{(a - x)^2}{x(a + x)}$.

9. $\frac{a(x - y)}{y(a + b)}$.

10. $\frac{x^2 - 3x + 2}{x^2}$.

11. $\frac{x^2 + y^2}{x}$.

12. $\frac{2m^2}{3n(m - n)}$.

13. $\frac{x^2 + y^2}{x^2 - y^2}$.

14. $\frac{x + y}{y}$.

15. $\frac{x^2 + y^2}{xy}$.

16. $\frac{x - 4}{x^2 - 2x + 8}$.

Art. 179

1. $\frac{3am^2}{7b^2n^2}$.

2. $\frac{2m^2}{3n(m - n)}$.

3. $\frac{a(x - y)}{y(a + b)}$.

4. $\frac{a^2}{b(a - b)}$.

5. $\frac{x^2 + y^2}{x}$.

6. $\frac{x^2 - 2a^2}{ax}$.

7. $\frac{ac - bd}{ac + bd}$.

8. $\frac{(x-1)(x-2)}{x^2}$.

9. $\frac{x-4}{x-5}$.

10. $\frac{1+x}{1+x^2}$.

11. a.

12. $\frac{1}{p^2+q^2}$.

13. $\frac{a^2+b^2}{2ab}$.

14. 1.

15. x.

16. $\frac{(x^2+y^2)^2}{x^4+y^4}$.

17. $\frac{(a^2+b^2)^2}{a^4+b^4}$.

Art. 191.

1. 10.
2. 2.
3. 10.
4. 12.
5. 3.
6. 3.
7. 9.
8. $2\frac{1}{3}$.
9. 2.
10. 4.
11. 1.
12. $3\frac{1}{3}$.
13. -28.
14. 1.
15. 5.
16. 60.
17. 41.
18. 7.
19. 8.
20. 9.
21. 4.
22. 5.
23. $\frac{1}{5}$.
24. 4.
25. 17.
26. 2.
27. $1\frac{1}{2}$.
28. $1\frac{8}{29}\frac{1}{9}$.
29. 5.
30. $2\frac{3}{13}$.

Art. 193.

1. 8.
2. 8.
3. 4.
4. 6.
5. $3\frac{3}{16}$.
6. $7\frac{8}{11}$.
7. 1.
8. $3\frac{3}{8}$.
9. $1\frac{1}{13}$.
10. 2.
11. 8.
12. $1\frac{3}{16}$
13. 7.
14. 4.
15. 4.
16. $\frac{8}{7}$.
17. 4.
18. 3.

Art. 195.

1. $\frac{a}{a-b+1}$.

2. $\frac{ab-de}{a-c}$.

3. $\frac{a^2-b^2}{2b}$.

4. $\frac{ab(c+d)}{a-b}$.

5. $\frac{c+d}{a+b}$.

6. $4b - 6c$.

7. $\frac{cd - ab}{a}$.

8. $\frac{c-b}{a-d}$.

9. $\frac{1}{ab}$.

10. $a - b$.

11. $a + b$.

12. $\frac{2ab}{2a+b}$.

13. $\frac{a(1-3a)}{3b}$.

14. $\frac{bm}{a+b+c}$.

15. $\frac{2ab}{a+b}$.

16. $\frac{a^2+ab+b^2}{a+b}$.

17. $\frac{b(a-b+c)}{a}$.

18. $\frac{a^2}{b-a}$.

19. $\frac{ac}{b}$.

20. $\frac{2ab}{a+b}$.

21. $4a$.

22. $\frac{8a}{25}$.

23. c.

Art. 196.

1. 11.
2. 42.
3. 11.
4. 19.
5. 13.
6. 15.
7. 5.
8. 27.
9. 5.
10. -3.
11. -23.
12. $5\frac{2}{3}\frac{0}{3}$.
13. $\frac{1}{3}$.
14. 5.
15. $2\frac{1}{2}$.
16. $63\frac{21}{65}$.
17. 2.
18. 3.
19. 10.
20. $\frac{2}{abc}$.
21. $\frac{c}{d}$.
22. 50.
23. 4.
24. 10

25. 5.

26. $\frac{b(c-a)}{a+c}$.

27. 10.

28. $\frac{abc+amn-bn}{bc+mn}$

29. $\frac{a-b}{ab}$.

30. $\frac{ab}{a-b}$.

31. $\frac{3(a+b)}{b-a}$.

32. $\frac{ac}{a-b}$.

33. 3.

34. 3.

35. $\frac{a^2+b^2}{2a}$.

36. $3\frac{3}{5}$.

37. 1.

38. $-1\frac{3}{23}$.

Art. 198.

1. $120.

2. 216.

3. 150 acres.

4. $1800.

5. Hat, $6; coat, $15.

6. House, $2600;
lot, $1175.

7. Father, $101.30;
son, $63.70.

8. Horse, $230;
carriage, $205;
sleigh, $165.

9. Wife, $8300;
son, $5000;
daughter, $2500.

10. 35, 50, and 70 miles.

11. 55 lb. at $.40;
45 lb. at $.50.

12. 6 days.

13. 12 days.

14. 8, 11, 14, and 17 yrs.

15. Overcoat, $29;
suit, $41; boots, $10.

16. $35; $19; $13.

17. Wheat, $1.13;
rye, $.92; oats, $.47.

18. 76 years; 38 years.

19. 48 years; 40 years.

20. 75; 100.

21. House, $5000;
lot, $3000.

22. $2.

23. 1st, $85; 2d, $170;
3d, $145.

24. $2800.

25. 120.

26. 21.

27. Watch, $145;
chain, $40.

28. House, $7200;
lot, $5600.

29. Cargo, $36400;
steamer, $91000.

30. $2000.

31. Copper, 420 oz.; tin, 85 oz.

32. 5, 17, and 26 oz.

33. 36.

34. A, \$43.75; B, \$49.25; C, \$62.

35. 48 feet.

36. A, 40; B, 29.

37. 36¢; 60¢.

38. A, \$50; B, \$20; C, \$30.

39. Father, \$2; son, \$1.65.

40. A, \$1500; B, \$500.

41. 57; 40.

42. \$60.

43. A, 40; B, 30.

44. \$12000.

45. 50 10-cent pieces; 20 25-cent pieces.

46. 85; 68; 119.

47. 20.

48. \$150; \$100; \$40.

49. Earns \$1000; spends \$800.

50. Working, 14; idle, 16.

51. \$.79.

52. \$4800.

53. 35 of 15-cent; 65 of 11-cent.

54. 22 men; 18 women; 50 children.

55. \$160.

56. 27; 24.

57. 484 miles.

58. 800.

59. \$2333$\frac{1}{3}$; \$2666$\frac{2}{3}$.

60. 824 bu. of each.

61. 7.

62. \$800; \$1200.

63. 120.

64. 22 3-cent; 17 5-cent pieces.

65. 31 ft.; 47 ft.; 22 ft.

66. 20 miles.

67. 12, corn; 9, potatoes.

68. 7 50-cent pieces; 9 25-cent "

69. 168 acres.

Art. 206.

1. $x = 4$, $y = 5$.
2. $x = 6$, $y = 3$.
3. $x = 8$, $y = 4$.
4. $x = 30$, $y = 25$.
5. $x = 9$, $y = 6$.
6. $x = 5$, $y = 4$.
7. $x = 3$, $y = 4$.
8. $x = 15$, $y = 9$.
9. $x = 2$, $y = 3$.
10. $x = 5$, $y = 2$.
11. $x = 3$, $y = 2$.
12. $x = 3\frac{1}{2}$, $y = 4$.
13. $x = 10$, $y = 7$.
14. $x = 19$, $y = 2$.
15. $x = 38\frac{1}{2}$, $y = 70$.
16. $x = 6$, $y = 12$.
17. $x = \frac{348}{157}$, $y = \frac{156}{157}$.
18. $x = 10$, $y = 5$.
19. $x = 12$, $y = 12$.
20. $x = 20$, $y = 20$.
21. $x = 10$, $y = 4$.
22. $x = 4$, $y = 9$.
23. $x = 5$, $y = 7$.
24. $x = 2\frac{1}{2}$, $y = 1$.
25. $x = 3$, $y = 2$.
26. $x = 63$, $y = 14$.

Art. 208.

1. $x = 10$, $y = 7$.
2. $x = 5$, $y = 2$.
3. $x = 2$, $y = -3$.
4. $x = 4$, $y = 12$.
5. $x = 3$, $y = 2$.
6. $x = 3\frac{1}{2}$, $y = 4$.
7. $x = \frac{1}{4}$, $y = \frac{3}{8}$.
8. $x = 19$, $y = 2$.
9. $x = 6$, $y = 12$.
10. $x = \frac{1}{4}$, $y = \frac{1}{3}$.
11. $x = 30$, $y = 25$.
12. $x = 56$, $y = 40$.
13. $x = 6$, $y = 5$.
14. $x = 3$, $y = 5$.
15. $x = 30$, $y = 20$.
16. $x = 6$, $y = 12$.
17. $x = 6$, $y = 8$.
18. $x = 36$, $y = 24$.
19. $x = 2$, $y = 37$.
20. $x = 7$, $y = 10$.
21. $x = 3$, $y = 2$.
22. $x = 18$, $y = 6$.
23. $x = 99$, $y = 15$.
24. $x = 4$, $y = 3$.
25. $x = 7$, $y = 4$.

Art. 211.

1. $x = 12, \quad y = 8.$
2. $x = 8, \quad y = 1.$
3. $x = 6, \quad y = 24.$
4. $x = 4, \quad y = 2.$
5. $x = 4, \quad y = 5.$
6. $x = 16, \quad y = 35.$
7. $x = 3, \quad y = 4.$
8. $x = 8, \quad y = 2.$
9. $x = 2, \quad y = 1.$
10. $x = 1, \quad y = -1.$
11. $x = 4, \quad y = 7.$
12. $x = 34, \quad y = 46.$
13. $x = 7, \quad y = 2.$
14. $x = 3, \quad y = 2.$
15. $x = 7, \quad y = 6.$
16. $x = 21, \quad y = 20.$
17. $x = 7, \quad y = 9.$
18. $x = 4, \quad y = 1.$
19. $x = \frac{1}{3}, \quad y = 2.$
20. $x = 11, \quad y = 12.$
21. $x = \frac{2}{7}, \quad y = \frac{4}{8}.$
22. $x = 4\frac{1}{2}, \quad y = 1\frac{1}{2}.$
23. $x = 1, \quad y = -11.$
24. $x = 11, \quad y = 13.$
25. $x = 3, \quad y = -\frac{1}{2}.$
26. $x = -2\frac{1}{2}, \quad y = 1\frac{1}{2}.$

Art. 213.

1. $x = \dfrac{cq - br}{aq - bp},$

   $y = \dfrac{ar - cp}{aq - bp}.$

2. $x = \dfrac{em + bn}{ae + bc},$

   $y = \dfrac{an - cm}{ae + bc}.$

3. $x = \dfrac{ac + b^2}{a^2 + b},$

   $y = \dfrac{ab - c}{a^2 + b}.$

4. $x = \dfrac{6a^2 - c}{a + 2b},$

   $y = \dfrac{ac + bc - 3a^3}{b(a + 2b)}.$

5. $x = \dfrac{ac + bd}{a^2 + b^2},$

   $y = \dfrac{bc - ad}{a^2 + b^2}.$

6. $x = \dfrac{bc}{a + b},$

   $y = \dfrac{ac}{a + b}.$

7. $x = a, \quad y = b.$
8. $x = a, \quad y = b.$
9. $x = \dfrac{ab}{a + b},$

   $y = \dfrac{ab}{a + b}.$

10. $x = b, \quad y = a.$

11. $x = \dfrac{ab^2c}{a^2 + b^2},$

$y = \dfrac{a^2bc}{a^2 + b^2}.$

12. $x = \dfrac{m^2 - n^2}{am - bn},$

$y = \dfrac{m^2 - n^2}{an - bm}.$

13. $x = \dfrac{ac}{a + b},$

$y = \dfrac{bc}{a + b}.$

14. $x = \dfrac{1}{a + b}, \quad y = 0.$

15. $x = a, \quad y = b.$

16. $x = a + b,$

$y = a - b.$

17. $x = (a + b)^2,$

$y = (a - b)^2.$

18. $x = \dfrac{c}{a + b},$

$y = \dfrac{c}{a + b}.$

19. $x = 1 - \dfrac{ab + c^2}{a + c},$

$y = \dfrac{ab + c^2}{b + c}.$

20. $x = (a + b)^2,$

$y = (a - b)^2.$

Art. 214.

1. $x = 10, \quad y = 5.$
2. $x = 6, \quad y = 1.$
3. $x = 16, \quad y = 35.$
4. $x = 2, \quad y = 3.$
5. $x = 1, \quad y = 2.$
6. $x = 3, \quad y = 2.$
7. $x = 2, \quad y = 5.$
8. $x = 1, \quad y = 3.$
9. $x = 17\frac{1}{2}, \; y = 115\frac{3}{5}.$
10. $x = 17\frac{1}{2}, \; y = 8\frac{2}{7}.$
11. $x = 8, \quad y = 2.$
12. $x = \frac{1}{2}, \quad y = \frac{1}{3}.$
13. $x = \frac{3}{4}, \quad y = \frac{4}{5}.$
14. $x = 1\frac{1}{2}, \; y = -2\frac{1}{2}.$
15. $x = 11, \quad y = -9.$
16. $x = 9, \quad y = 123\frac{1}{2}.$
17. $x = -12, \quad y = 50.$
18. $x = a, \quad y = b.$
19. $x = \dfrac{1}{m + n}, \quad y = 0$
20. $x = \dfrac{a}{bc}, \quad y = \dfrac{a + 2b}{c}.$

Art. 211.

1. $x = 12, \quad y = 8.$
2. $x = 8, \quad y = 1.$
3. $x = 6, \quad y = 24.$
4. $x = 4, \quad y = 2.$
5. $x = 4, \quad y = 5.$
6. $x = 16, \quad y = 35.$
7. $x = 3, \quad y = 4.$
8. $x = 8, \quad y = 2.$
9. $x = 2, \quad y = 1.$
10. $x = 1, \quad y = -1.$
11. $x = 4, \quad y = 7.$
12. $x = 34, \quad y = 46.$
13. $x = 7, \quad y = 2.$
14. $x = 3, \quad y = 2.$
15. $x = 7, \quad y = 6.$
16. $x = 21, \quad y = 20.$
17. $x = 7, \quad y = 9.$
18. $x = 4, \quad y = 1.$
19. $x = \frac{1}{3}, \quad y = 2.$
20. $x = 11, \quad y = 12.$
21. $x = \frac{2}{7}, \quad y = \frac{5}{8}.$
22. $x = 4\frac{1}{2}, \quad y = 1\frac{1}{2}.$
23. $x = 1, \quad y = -11.$
24. $x = 11, \quad y = 13.$
25. $x = 3, \quad y = -\frac{1}{2}.$
26. $x = -2\frac{1}{2}, \quad y = 1\frac{1}{2}.$

Art. 213.

1. $x = \dfrac{cq - br}{aq - bp},$

   $y = \dfrac{ar - cp}{aq - bp}.$

2. $x = \dfrac{em + bn}{ae + bc},$

   $y = \dfrac{an - cm}{ae + bc}.$

3. $x = \dfrac{ac + b^2}{a^2 + b},$

   $y = \dfrac{ab - c}{a^2 + b}.$

4. $x = \dfrac{6a^2 - c}{a + 2b},$

   $y = \dfrac{ac + bc - 3a^3}{b(a + 2b)}.$

5. $x = \dfrac{ac + bd}{a^2 + b^2},$

   $y = \dfrac{bc - ad}{a^2 + b^2}.$

6. $x = \dfrac{bc}{a + b},$

   $y = \dfrac{ac}{a + b}.$

7. $x = a, \quad y = b.$
8. $x = a, \quad y = b.$
9. $x = \dfrac{ab}{a + b},$

   $y = \dfrac{ab}{a + b}.$

10. $x = b, \quad y = a.$

11. $x = \dfrac{ab^2c}{a^2 + b^2},$

$y = \dfrac{a^2bc}{a^2 + b^2}.$

12. $x = \dfrac{m^2 - n^2}{am - bn},$

$y = \dfrac{m^2 - n^2}{an - bm}.$

13. $x = \dfrac{ac}{a + b},$

$y = \dfrac{bc}{a + b}.$

14. $x = \dfrac{1}{a + b}, \quad y = 0.$

15. $x = a, \quad y = b.$

16. $x = a + b,$

$y = a - b.$

17. $x = (a + b)^2,$

$y = (a - b)^2.$

18. $x = \dfrac{c}{a + b},$

$y = \dfrac{c}{a + b}.$

19. $x = 1 - \dfrac{ab + c^2}{a + c},$

$y = \dfrac{ab + c^2}{b + c}.$

20. $x = (a + b)^2,$

$y = (a - b)^2.$

Art. 214.

1. $x = 10, \quad y = 5.$

2. $x = 6, \quad y = 1.$

3. $x = 16, \quad y = 35.$

4. $x = 2, \quad y = 3.$

5. $x = 1, \quad y = 2.$

6. $x = 3, \quad y = 2.$

7. $x = 2, \quad y = 5.$

8. $x = 1, \quad y = 3.$

9. $x = 17\frac{1}{2}, \; y = 115\frac{1}{2}.$

10. $x = 17\frac{1}{2}, \; y = 8\frac{1}{2}.$

11. $x = 8, \quad y = 2.$

12. $x = \frac{1}{2}, \quad y = \frac{1}{3}.$

13. $x = \frac{3}{4}, \quad y = \frac{2}{5}.$

14. $x = 1\frac{1}{2}, \; y = -2\frac{1}{2}.$

15. $x = 11, \quad y = -9.$

16. $x = 9, \quad y = 123\frac{1}{2}.$

17. $x = -12, \quad y = 50.$

18. $x = a, \quad y = b.$

19. $x = \dfrac{1}{m + n}, \quad y = 0$

20. $x = \dfrac{a}{bc}, \quad y = \dfrac{a + 2b}{c}.$

29. A, \$1500; B, \$702; C, \$1026.

30. A, \$400; B, \$640; C, \$780.

31. 1st to 2d, 52 mi.; 2d to 3d, 45 mi.; 1st to 3d, 37 mi.

32. A, \$6$\frac{1}{2}$; B, \$6$\frac{3}{4}$; bill, \$7$\frac{1}{4}$.

33. 3 hrs., 3$\frac{1}{2}$ hrs., 4$\frac{1}{2}$ hrs., 1 hr. 12$\frac{3}{5}\frac{3}{7}$ min.

34. 20¢, 24¢, 18¢.

35. 972.

36. \$50, \$100, \$120.

37. 2 mi. up, 4 mi. down, 3 mi. level.

38. A in 24 da., B in 48 da.

39. 5, 8, 13.

40. A in 20 da., B in 30 da., C in 60 da., all in 10 da.

41. First horse, \$120; second horse, \$150.

42. Son, \$4500; daughter, \$3000; mother, \$3750.

Art. 223.

1. $a^3b^3x^6$.

2. $-x^6y^3$.

3. $x^{16}y^4z^8$.

4. $243a^{15}x^{10}$.

5. $-32m^{15}n^5$.

6. $81m^4n^8x^4$.

7. $25m^8n^6y^4$.

8. $-343a^9m^3n^6$.

9. $256a^8b^{10}m^{18}$.

10. $1000x^6y^9z^{12}$.

11. $625a^8b^{12}c^{32}d^{16}$.

12. $-1024a^{25}m^{15}n^{20}x^{10}$.

13. $\frac{1}{25}m^6$, $\frac{1}{9}m^4n^2$, $16a^4b^4c^6$.

14. $\frac{1}{27}a^6b^3$, $\frac{1}{64}x^9y^6z^3$, $125a^3m^{12}x^6$.

15. $100a^6b^2x^4$, $225x^2y^2z^8$.

16. $-\frac{125}{1331}x^6y^3z^9$, $\frac{8}{1331}m^{12}n^6y^9$.

17. $3\frac{1}{2}a^5m^{10}n^{10}$, $-2\frac{1}{4}\frac{1}{3}x^{10}y^5z^{15}$.

18. $\frac{1}{16}p^4q^2$, $\frac{1}{25}a^2x^4y^2$, $\frac{1}{25}a^2m^4$.

19. $\frac{1}{81}a^3m^{12}n^4$, $\frac{81}{256}a^4b^8c^8x^4$.

20. $\frac{64}{27}x^9y^3z^{12}$, $\frac{729}{1000}m^{12}n^9y^6$.

21. $\frac{4x^4}{9y^2}$, $\frac{9a^2x^4}{4y^6}$, $\frac{x^4y^4}{9a^2z^2}$.

22. $\frac{64a^6x^{12}}{125y^6z^3}$, $-\frac{27a^9x^3y^3}{512m^6n^9}$.

23. $\frac{81a^8b^{16}c^4}{625x^4y^8z^8}$, $\frac{32m^{12}n^8}{81x^8y^{12}z^{12}}$.

Art. 225.

1. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$.

2. $a^2 + 4b^2 + 9c^2 + 4ab + 6ac + 12bc$.

3. $4m^2 + 9n^2 + 25y^2 - 12mn + 20my - 30ny$.

4. $9x^2 + 4y^2 + 16z^2 - 12xy - 24xz + 16yz$.

5. $81x^2 + 9y^2 + z^2 - 54xy + 18xz - 6yz$

6. $x^4 + 4x^3 + 6x^2 + 4x + 1$.

7. $a^4 + b^6 + c^4 + d^2 + 2a^2b^3 - 2a^2c^2 + 2a^2d - 2b^3c^2 + 2b^3d - 2c^2d$.

8. $m^4 + 8m^3 + 18m^2 + 8m + 1$.

9. $a^2 + b^2 + c^2 + d^2 + 2ab + 2ac - 2ad + 2bc - 2bd - 2cd$.

10. $x^6 - 6x^5 + 13x^4 - 22x^3 + 34x^2 - 20x + 25$.

11. $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1$.

12. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

13. $\frac{9}{16}x^2 + \frac{3}{4}y^2 + \frac{49}{16}z^2 - \frac{3}{4}xy + \frac{21}{8}xz - \frac{7}{4}yz$.

14. $4x^4 + 4x^3 - 27x^2 - 14x + 49$.

15. $x^4 + 2x^3 + 3x^2 + 2x + 1$.

16. $\frac{1}{4}a^2 + a^4b^2 + \frac{1}{4}b^2 + a^3b + \frac{1}{2}ab + a^2b^2$.

17. $m^4 + m^3 + \frac{1}{4}m^2 + \frac{1}{2}m + \frac{1}{4}$.

18. $25x^2 + 9y^4 - .0625z^2 + 30xy^2 - 2.5xz - 1.5y^2z$.

Art. 227.

1. $a^3 + 3a^2b + 3ab^2 + b^3$.

2. $8x^3 + 12x^2y + 6xy^2 + y^3$.

3. $27m^3 - 54m^2n + 36mn^2 - 8n^3$.

4. $8a^3 - 36a^2b + 54ab^2 - 27b^3$.

5. $27a^3 + 108a^2b + 144ab^2 + 64b^3$.

6. $64a^3 - 144a^2y + 108ay^2 - 27y^3$.

7. $m^3 - 6m^2n + 12mn^2 - 8n^3$.

8. $a^3 + 9a^2b + 27ab^2 + 27b^3$.

9. $8a^3 - 12a^2b + 6ab^2 - b^3$.

10. $1 + 3x + 3x^2 + x^3$.

11. $1 - 3x^2 + 3x^4 - x^6$.

12. $x^3 - 6x^2y + 12xy^2 - 8y^3$.

13. $a^3 + 15a^2 + 75a + 125$.

14. $\frac{1}{8}x^3 - \frac{3}{8}x^2y + \frac{3}{8}xy^2 - \frac{1}{8}y^3$.

15. $1 - 3mn + 3m^2n^2 - m^3n^3$.

16. $8a^3 + 12a^2c + 6ac^2 + c^3$.

17. $\frac{1}{64}x^3 + \frac{3}{32}x^2y + \frac{3}{16}xy^2 + \frac{1}{8}y^3$.

18. $64x^3 + 48x^2y + 12xy^2 + y^3$.

19. $\frac{8}{27}a^3 + \frac{4}{3}a^2b + 2ab^2 + b^3$.

20. $x^3 - x^2y + \frac{1}{2}xy^2 - \frac{1}{27}y^3$.

21. $a^6 + 3a^4b^2 + 3a^2b^4 + b^6$.

22. $x^6 - 6x^4y + 12x^2y^2 - 8y^3$.

23. $m^3 - 3m^2n^2 + 3mn^4 - n^6$.

24. $27a^6 + 27a^4b^2 + 9a^2b^4 + b^6$.

25. $x^9 + 6x^8y + 12x^7y^2 + 8x^6y^3$.

26. $x^6 + 3x^5y + 3x^4y^2 + x^3y^3$.

27. $8x^6 - 36x^5y + 54x^4y^2 - 27x^3y^3$.

28. $27a^3b^3 - 54a^2b^4 + 36ab^5 - 8b^6$.

Art. 229.

1. $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

2. $x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3$.

3. $m^3 - 3m^2n + 3mn^2 - n^3 - 3m^2r + 6mnr - 3n^2r + 3mr^2 - 3nr^2 - r^3$.

4. $a^3 + 6a^2b + 12ab^2 + 8b^3 + 3a^2c + 12abc + 12b^2c + 3ac^2 + 6bc^2 + c^3$.

5. $x^3 - 6x^2y + 12xy^2 - 8y^3 + 3x^2z - 12xyz + 12y^2z + 3xz^2 - 6yz^2 + z^3$.

6. $x^3 + 6x^2y + 12xy^2 + 8y^3 + 9x^2z + 36xyz + 36y^2z + 27xz^2 + 54yz^2 + 27z^3$.

7. $a^3 - 6a^2b + 12ab^2 - 8b^3 - 9a^2c + 36abc - 36b^2c + 27ac^2 - 54bc^2 - 27c^3$.

8. $8a^3 - 12a^2m + 6am^2 - m^3 + 24a^2n - 24amn + 6m^2n + 24an^2 - 12mn^2 + 8n^3$.

9. $8p^3 - 36p^2q + 54pq^2 - 27q^3 - 60p^2r + 180pqr - 135q^2r + 150pr^2 - 225qr^2 - 125r^3$.

10. $a^3 - 15a^2y + 75ay^2 - 125y^3 + 3a^2z - 30ayz + 75y^2z + 3az^2 - 15yz^2 + z^3$.

11. $a^6 + 3a^4x^2 + 3a^2x^4 + x^6 - 3a^4y^2 - 6a^2x^2y^2 - 3x^4y^2 + 3a^2y^4 + 3x^2y^4 - y^6$.

12. $27x^6 + 27x^4y^2 + 9x^2y^4 + y^6 + 54x^4z + 36x^2y^2z + 6y^4z + 36x^2z^2 + 12y^2z^2 + 8z^3$.

13. $125x^8 - 150x^4 + 135x^5 - 68x^6 + 27x^7 - 6x^8 + x^9$.

14. $a^6 + 3a^5b + 6a^4b^2 + 7a^3b^3 + 6a^2b^4 + 3ab^5 + b^6$.

15. $x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6$.

16. $x^6 + 6x^5 + 21x^4 + 44x^3 + 63x^2 + 54x + 27$.

17. $8x^6 + 36x^5y + 66x^4y^2 + 63x^3y^3 + 33x^2y^4 + 9xy^5 + y^6$.

18. $27a^6 - 54a^5y + 171a^4y^2 - 188a^3y^3 + 285a^2y^4 - 150ay^5 + 125y^6$.

19. $m^6 - 15m^5n + 84m^4n^2 - 215m^3n^3 + 252m^2n^4 - 135mn^5 + 27n^6$.

20. $27a^{12} + 135a^{10}y^2 + 171a^8y^4 - 55a^6y^6 - 114a^4y^8 + 60a^2y^{10} - 8y^{12}$.

21. $x^{12} - 3x^{10}y^2 + 6x^8y^4 - 7x^6y^6 + 6x^4y^8 - 3x^2y^{10} + y^{12}$.

22. $1 + 3p^2 - 5p^6 + 3p^{10} - p^{12}$.

23. $8x^6 - 36x^5y + 66x^4y^2 - 63x^3y^3 + 33x^2y^4 - 9xy^5 + y^6$.

24. $x^{12} - 9x^{10}y + 24x^8y^2 - 9x^6y^3 - 24x^4y^4 - 9x^2y^5 - y^6$.

25. $x^6 - 6x^5y + 9x^4y^2 + 4x^3y^3 - 9x^2y^4 - 6xy^5 - y^6$.

26. $x^9 - 6x^8y^2 + 12x^7y^4 - 8x^6y^6 + 3x^6y^8 - 12x^5y^5 + 12x^4y^7 + 3x^3y^6 - 6x^2y^8 + y^9$.

27. $x^6 - 15x^5 + 93x^4 - 305x^3 + 558x^2 - 540x + 216$.

28. $x^{12} - 27x^{10}y + 267x^8y^2 - 1161x^6y^3 + 2136x^4y^4 - 1728x^2y^5 + 512y^6$.

Art. 232.

1. $a^2 + 2ax + x^2$.
2. $m^2 - 2mn + n^2$.
3. $a^3 + 3a^2b + 3ab^2 + b^3$.
4. $p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$.
5. $4a^2 + 4ax + x^2$.
6. $x^3 - 6x^2y + 12xy^2 - 8y^3$.
7. $8a^3 + 36a^2m + 54am^2 + 27m^3$.
8. $27x^3 + 135x^2 + 225x + 125$.
9. $8x^3 + 60x^2y + 150xy^2 + 125y^3$.
10. $625x^4 + 500x^3 + 150x^2 + 20x + 1$.
11. $1 + 4x + 6x^2 + 4x^3 + x^4$.
12. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.
13. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.
14. $32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$.
15. $x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$.
16. $243x^5 + 405x^4y + 270x^3y^2 + 90x^2y^3 + 15xy^4 + y^5$.
17. $m^5 + 25m^4n + 250m^3n^2 + 1250m^2n^3 + 3125mn^4 + 3125n^5$.
18. $32p^5 - 240p^4q + 720p^3q^2 - 1080p^2q^3 + 810pq^4 - 243q^5$.
19. $a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4$.
20. $16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$.
21. $\frac{8}{64}x^3 + \frac{8}{16}x^2y + \frac{2}{4}xy^2 + y^3$.
22. $x^3 - \frac{3}{2}x^2y + \frac{3}{4}xy^2 - \frac{1}{8}y^3$.
23. $\frac{8}{27}a^3 - a^2x + \frac{9}{8}ax^2 - \frac{8}{64}x^3$.
24. $a^6 + 3a^4b^2 + 3a^2b^4 + b^6$.

25. $x^3 + 3x^2y^2 + 3xy^4 + y^6$.

26. $\frac{1}{16}m^4 - \frac{1}{2}m^3n^2 + \frac{3}{2}m^2n^4 - 2mn^6 + n^8$.

27. $a^5x^5 + 5a^4x^4 + 10a^3x^3 + 10a^2x^2 + 5ax + 1$.

28. $27a^3 + 27a^2xy + 9ax^2y^2 + x^3y^3$.

29. $\frac{81}{16}x^4 + 9x^3y + 6x^2y^2 + \frac{16}{9}xy^3 + \frac{16}{81}y^4$.

30. $\frac{16}{81}x^4 - \frac{8}{9}x^3y + \frac{3}{2}x^2y^2 - \frac{9}{8}xy^3 + \frac{81}{256}y^4$.

Art. 244.

1. $\pm 5ab^2$.

2. $\pm 6x^2y^3$.

3. $\pm 12m^2n^3$.

4. $\pm 9a^2b^3c^6$.

5. $5xy^2z^3$.

6. $\pm 3xy^2z^3$.

7. $2pq^2r^3$.

8. $\pm 4a^2bc^3$.

9. $-3m^2n$.

10. $-2am^2x^3$.

11. $\pm 5a^2b^3c$.

12. $3a^2b^3c^4$.

13. $8x^2yz^5$.

14. $11ab^2x^3$.

15. $-3ax^2y^4$.

16. $-5m^2n^3p^4$.

17. $\pm \frac{3}{5}ab^2c^3$.

18. $\frac{3}{4}ab^2c^3$.

19. $-\frac{4ax^2y}{5mn^4r^2}$.

20. $-\frac{5}{6}a^2bx^4$.

21. $\frac{7}{16}ab^4c^3$.

22. $\pm \frac{2}{5}x^2y^3z^4$.

23. $\frac{2}{3}m^4n^2r^3$.

24. $\frac{6ab^2c^5}{5x^2y^3z^4}$.

Art. 246.

1. $a + b$.

2. $a + 3$.

3. $x - y$.

4. $m - n$.

5. $a + 2b$.

6. $m + 3n$.

7. $a - 4b$.

8. $x + 7y$.

9. $ax + 1$.

10. $ab - 3$.

11. $1 + 3mn$.
12. $2x - 3y$.
13. $10m^2 - 9n^2$.
14. $8ab - 7x^2$.
15. $11x^2y + 5z^2$.
16. $5ax^2 - 2y^3$.
17. $a^2 - a + 1$.
18. $2y^2 + 3y - 1$.
19. $2x^2 - x + 1$.
20. $2a^2 - 3ay + 4y^2$.
21. $5m^2 - 3mn + 4n^2$.
22. $a^3 - 3a^2b + 3ab^2 - b^3$.
23. $x^3 - 3x^2 + 4x - 5$.
24. $2x^3 - 3x^2y - 5xy^2 + 7y^3$.
25. $m^3 - 5m^2n - 3mn^2 - 6n^3$.

Art. 249.

1. 24.
2. 28.
3. 37.
4. 45.
5. 67.
6. 64.
7. 93.
8. 97.
9. 107.
10. 123.
11. 145.
12. 305.
13. 964.
14. 327.
15. 945.
16. 409.
17. 617.
18. 1.324.
19. 2173.
20. 1507.
21. $\frac{8}{33}$.
22. $\frac{6}{11}$.
23. $\frac{137}{143}$.
24. $\frac{141}{311}$.
25. 4309.
26. 6089.
27. 42.87.
28. 4.305.
29. .7508.
30. 987.6.
31. 4.5825+.
32. .2236+.
33. .0948+.
34. 6.4031+.
35. 10.6301+.
36. 24.9799+.
37. .7905+.
38. .9574+.
39. .8944+.
40. .9128+.
41. 1.9364+.
42. 2.7735+.

Art. 251.

1. $x - y$.
2. $m + n$.
3. $x + 3$.
4. $2a + 3$.

5. $3x - 2$.
6. $x^2 - 2$.
7. $3y^2 - 5$.
8. $ax + 2$.
9. $5x^2 - 1$.
10. $2x^2 - 5y^3$.
11. $5a^2b - 3$.
12. $x^2 - 2x + 1$.
13. $x^2 - 3x + 2$.
14. $2x^2 - 3ax + 4a^2$.
15. $2m^2 + 3mn - n^2$.
16. $3a^2 - 5a - 2$.
17. $6a^4 - 4a^2 - 3$.
18. $2x^2 - 3xy + 5y^2$.
19. $3a^2 - 4a - 7$.
20. $2x^2 + 4ax - 3a^2$.

Art. 255.

1. 17.
2. 41.
3. 68.
4. 97.
5. 123.
6. 345.
7. 678.
8. 497.
9. 805.
10. 409.
11. 906.
12. 709.
13. 123.4.
14. 56.78.
15. 3.245.
16. .8264.
17. 70.65.
18. 9.208.
19. 30.07.
20. 8905.

Art. 259.

1. a^2.
2. $x^{\frac{1}{2}}y^{\frac{1}{2}}$.
3. $x^{\frac{1}{6}}y^{\frac{1}{6}}$.
4. $mn^{\frac{1}{2}}$.
5. x.
6. $a^{\frac{7}{4}}$.
7. $a^{\frac{1}{4}}$.
8. $18a^2b^2$.
9. $6x^2$.
10. $4x^{-3}$.
11. $n^{\frac{1}{2}}$.
12. $ab^{-\frac{1}{4}}$.
13. x.
14. $a^{2n}b^n$.
15. $x^{-\frac{n}{2}}y^{n+1}$.
16. $a^{\frac{2n}{4}}$.
17. $a^{\frac{2}{n}}b^{\frac{2}{3}}$.
18. $a^{-2}b^{-3}$.
19. m^{3n}.
20. x^{-3}.
21. x^{-6}.
22. y.

Art. 260.

1. $a - b$.

2. $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

3. $m - m^{\frac{1}{2}}n^{\frac{1}{2}} + m^{\frac{1}{2}}n^{\frac{1}{2}} - n$.

4. $x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} - y^{-1}$.

5. $a^{-\frac{1}{2}} - a^{-\frac{1}{2}}b^{-\frac{1}{2}} + a^{-\frac{1}{2}}b^{-\frac{1}{2}} - b^{-\frac{1}{2}}$.

6. $x^{-\frac{1}{2}} + 2x^{-\frac{1}{2}}y^{\frac{1}{2}} + y$.

7. $x^{-1} + x^2y^2 - x^{-3}y^{-3} - y^{-1}$.

8. $p^{\frac{1}{2}} + p^{\frac{1}{2}}q^{\frac{1}{2}} - p^{\frac{1}{2}}q^{-\frac{1}{2}} - q^{\frac{1}{2}}$.

9. $x^{-4} + x^{-2}y^{-2} + y^{-4}$.

10. $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$.

11. $x^{-\frac{1}{2}} - x^{-\frac{1}{2}}y + x^{-1}y^{\frac{1}{2}} - 2x^{-\frac{1}{2}}y^2 + y^{\frac{1}{2}}$.

12. $x - 2x^{\frac{1}{2}} + 1$.

13. $x^{-\frac{7}{2}} - 109x^{-\frac{5}{2}} - 12x^{-1} - 56$.

14. $5x^{-2} + 303x^{-\frac{5}{3}} - 378x^{-\frac{4}{3}} - 11x^{-\frac{1}{3}} + 2$.

15. $.02x^{4m} + 1.28x^{-4r}$.

16. $25x^{-6} + 176x^{-4} - 266x^{-3} + 1$.

17. $\frac{1}{6}x^{-2} + .0054y^{\frac{1}{3}}$.

18. $\frac{25}{6}m^{\frac{5}{3}} - \frac{31}{240}m^{\frac{2}{3}} - \frac{1}{100}$.

19. $\frac{2}{5}x^{-\frac{1}{5}} - \frac{12}{25}x^{-\frac{2}{5}}y^{-2} + \frac{18}{25}y^{-4}$.

20. $84x^2 - 437x^{\frac{1}{2}} + 1$.

Art. 262.

1. $3a^{-2}$.
2. $4x^{-4}$.
3. $2x^{5}$.
4. $b^{-\frac{1}{4}}$.
5. $x^{\frac{1}{2}}y^{\frac{1}{3}}$.
6. xy^{3}.
7. $5ab$.
8. $2x^{-n}y$.
9. $a^{\frac{1}{2}}bc^{-3}x$.
10. $8b^{\frac{3}{2}}c^{-1}dx$.
11. $a^{-\frac{1}{2}}b^{-\frac{1}{3}}c^{-2}$.
12. xyz^{-1}.
13. $5m^{2}n^{-2}x^{-2}$.
14. $\frac{1}{2}xyz^{-1}$.
15. $\frac{2}{3}mn^{-1}$.
16. $2a^{-3}b^{-2}x^{\frac{1}{3}} - 3b^{-1}x^{-\frac{1}{4}}$.
17. $\frac{2}{3}x^{n-1}yz^{-1} + \frac{5}{6}x^{-2}y^{-\frac{1}{3}}z^{-1}$.
18. $\frac{2}{3}a^{-2}b^{n-1} - \frac{5}{6}ab^{-\frac{1}{3}}c^{-2}x^{-2}$.
19. $1\frac{1}{2}am^{-\frac{3}{4}}n - 2anx$.
20. $\frac{2}{3}ax^{\frac{3}{4}}z^{\frac{3}{4}+n} + \frac{3}{5}x^{2}y^{\frac{1}{2}}z^{n}$.
21. $4a^{-\frac{3}{5}}b^{-2}c^{-\frac{1}{3}} - 5a^{-\frac{3}{5}}b^{n-1}c^{-2}$.
22. $\frac{2}{3}x^{n+1}y^{r-1}z + \frac{1}{2}y^{-3}z^{\frac{1}{2}}$.
23. $\frac{1}{2}x^{n+\frac{1}{3}}z^{-\frac{5}{4}} - 1\frac{1}{5}xy^{2}z$.

Art. 263.

1. $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.
2. $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.
3. $x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{\frac{1}{3}} + x^{\frac{1}{4}}y^{\frac{1}{2}} - y^{\frac{1}{3}}$.
4. $m^{\frac{1}{3}} - m^{\frac{1}{4}}n^{\frac{1}{4}} + n^{\frac{1}{3}}$.
5. $p^{\frac{2}{3}} + p^{\frac{1}{3}}q^{\frac{1}{3}} + q$.
6. $x^{-2} + x^{-\frac{3}{4}}y^{\frac{1}{3}} + x^{-1}y^{\frac{2}{3}} + x^{-\frac{1}{2}}y + y^{\frac{4}{3}}$.

7. $a^{\frac{1}{3}} + a^{\frac{2}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}$.

8. $x - 2x^{\frac{1}{2}}y^{-\frac{1}{2}} + 4y^{-\frac{2}{3}}$.

9. $x^{\frac{n}{3}} + 3$.

10. $x^{-3} - 5x^{-2} + 5x^{-1} - 3$.

11. $a^{-\frac{4}{3}} - 3a^{-\frac{3}{3}} + 9a^{-\frac{2}{3}} - 27a^{-\frac{1}{3}} - 1$.

12. $m^n + m^{\frac{n}{2}}x^{\frac{p}{2}} + x^p$.

13. $2x^2 + 3x^{\frac{3}{2}} + 5x - 2$.

14. $2x - 6x^{\frac{3}{4}} + 3x^{\frac{1}{2}} - 1$.

15. $x^{\frac{3}{4}} + 5x^{\frac{1}{2}} + 4x^{\frac{1}{4}} - 3$.

16. $x^{\frac{1}{3}} + 4x + 13x^{\frac{2}{3}} + 42x^{\frac{1}{3}} + 28$.

17. $2a^{-\frac{3}{5}} - 8a^{-\frac{2}{5}} + 8a^{-\frac{1}{5}} - 14$.

18. $4x^{-\frac{4}{3}} + 10x^{-1} + 23x^{-\frac{2}{3}} + 5x^{-\frac{1}{3}} + 1$.

19. $x^{\frac{4}{3}} + 3x + 5x^{\frac{2}{3}} + 7x^{\frac{1}{3}} + 9$.

20. $x - 3x^{\frac{1}{2}} + 6 - 3x^{-\frac{1}{2}} - 6x^{-1}$.

Art. 265.

1. $x^6y^{-9}z^3$.

2. $16a^2b^3c^{-2}$.

3. $9m^{2n}x^{-1}y^{-2}$.

4. $25b^{-4}c^{3n-2}d^{2n+2}$.

5. $1000a^{-2}bc^{-4}$.

6. $64x^{-6}y^{-2}z^{2m}$.

7. $3a^{-2}b^{-3}x^4$.

8. $4a^{\frac{1}{3}}b^{-2}c^{\frac{1}{3}}$.

9. $4x^{-\frac{1}{3}}y^{4m}z^{-\frac{4m}{3}}$.

10. $5p^mq^{-n}r^{-\frac{n}{2}}$.

11. $\frac{9}{16}a^{\frac{3}{4}}m^{-3}y^{-\frac{3}{4}}$.

12. $49a^{-3}x^{\frac{3}{2}}y^5$.

13. $3x^py^{-\frac{2p}{3}}z^n$.

14. $16a^{4m}b^nc^{-\frac{m}{n}}$.

15. $5x^{\frac{3}{4}}y^{-1}z^{\frac{3}{4}}$.

16. $10000x^{-1}y^{4}z^{2m}$.

17. $144a^{-1}b^{-3}m^{3n}$.

18. $16x^{\frac{1}{m}}y^{-\frac{n}{m}}z^{\frac{2}{m}}$.

Art. 267.

1. $a^{\frac{1}{2}}x^{\frac{1}{2}}$.
2. $m^{\frac{3}{4}}$.
3. $m^{\frac{1}{2}}n^{\frac{3}{4}}$.
4. $a^{\frac{1}{2}}x^{\frac{3}{4}}$.
5. $a^{2}x^{\frac{1}{2}}$.
6. $a^{\frac{1}{2}}b^{-\frac{1}{2}}$.
7. $x^{\frac{m}{n}}y$.
8. $a^{\frac{1}{2}}b^{\frac{m}{n}}$.
9. $a^{\frac{1}{2}}(x-y)^{\frac{1}{2}}$.
10. $a^{2}(x+y)^{\frac{1}{2}}$.
11. $a^{-4}(m-n)^{-6}$.
12. $a^{n}b^{\frac{n}{m}}(x^{2}+y)$.
13. $a^{\frac{1}{2}}b^{\frac{1}{2}}x^{\frac{3}{4}}y^{\frac{1}{2}}$.
14. $a^{\frac{m}{n}}b^{\frac{2}{p}}x^{\frac{1}{2}}$.
15. $5a^{\frac{3}{4}}bm^{\frac{1}{2}}x^{-\frac{3}{4}}$.
16. $15(a-b)^{\frac{1}{2}}$.

Art. 268.

1. $\sqrt{a}$.
2. $\sqrt{xy}$.
3. $\sqrt[3]{ax^{2}}$.
4. $\sqrt[4]{m^{3}n}$.
5. $\sqrt{a} \times \sqrt[3]{b}$.
6. $\sqrt[3]{a^{2}c} \times \sqrt{b}$.
7. $2a\sqrt{b}$.
8. $5x\sqrt{a}$.
9. $3\sqrt[3]{a^{2}b}$.
10. $\sqrt{5ab^{3}}$.
11. $4\sqrt[4]{a^{3}b}$.
12. $2\sqrt[5]{xy^{2}z^{3}}$.

Art. 271.

1. $\frac{x}{a^{2}}$.
2. $\frac{ax^{2}}{m^{2}n^{3}}$.
3. $\frac{5a^{3}b^{2}}{7x^{4}y^{n}}$.
4. $\frac{y}{ax}$.
5. $\frac{ab}{amb+m}$.
6. $9a^{5}b$.
7. $\frac{15x^{3}y}{a^{2}b^{3}}$.
8. $\frac{1}{a+1}$.
9. $\frac{ab^{2}}{a+b^{2}}$.
10. $a^{2}-4a+4$.

11. $\frac{x^2+y^2}{xy(x-y)}$.

12. $\frac{a(a+b^2)}{b(a^2+b)}$.

13. $\frac{1+x+x^2}{x^2}$.

14. ab.

15. $\frac{b}{a}$.

16. xy.

17. $\frac{a^m b^m}{a^m+b^m}$.

18. $\frac{x^2y^2}{x^2+y^2}$.

19. $5ax-3x-5a+3$.

20. $\frac{2xy-x^2-y^2}{xy}$.

Art. 273.

1. $\sqrt{x}$.

2. $3\sqrt[3]{a}$.

3. $5\sqrt{ax}$.

4. $3\sqrt[3]{mn}$.

5. $\frac{6\sqrt{x}}{\sqrt[3]{a^2}}$.

6. $\frac{11\sqrt{m}\times\sqrt[3]{x}}{\sqrt{n}}$.

7. $\frac{5\sqrt[3]{a^2y}}{\sqrt[4]{x^3}}$.

8. $\frac{7\sqrt[3]{z^2}}{\sqrt[3]{xy}}$.

9. $\frac{3\sqrt{xy}}{4\sqrt[3]{ab}}$.

10. $6mn$.

11. $\frac{3\sqrt[3]{x}}{2a\sqrt{b}}$.

12. $\frac{4\sqrt{an}\times\sqrt[3]{x}}{5\sqrt{m}}$.

13. $\frac{\sqrt[3]{x}\times\sqrt{y}}{\sqrt{a}\times\sqrt[3]{b}}=\sqrt[6]{\frac{x^2y^3}{a^3b^2}}$.

14. $\frac{\sqrt{ax}}{\sqrt[3]{by}}$.

15. $9\sqrt{a+b}$.

16. $3\sqrt[3]{(a-b)^2}$.

17. $5a\sqrt[4]{(x+3)^3}$.

18. $17\sqrt{x}\times\sqrt[3]{(a-2)^2}$.

19. $\frac{3\sqrt[3]{m^2}\times\sqrt{x-1}}{4\sqrt{a}\times\sqrt[3]{x+1}}$.

20. $\frac{13\sqrt[3]{a^2}}{\sqrt[3]{(x^2+y)^2}}$.

Art. 275.

1. 6.
2. 8.
3. 15.
4. 42.
5. 72.
6. 24.
7. 160.
8. 288.
9. 24.
10. 300.
11. 924.
12. $28ax^2$.
13. 900.
14. 13824.
15. 1728000.
16. 30.
17. 1650.
18. 120.

Art. 281.

1. $2\sqrt{3}$.
2. $2\sqrt{5}$.
3. $3\sqrt{2}$.
4. $3\sqrt{3}$.
5. $4\sqrt{2}$.
6. $4\sqrt{3}$.
7. $5\sqrt{2}$.
8. $3\sqrt{5}$.
9. $3a\sqrt{7a}$.
10. $3a\sqrt{6b}$.
11. $6a^2b\sqrt{2b}$.
12. $5xy\sqrt{3y}$.
13. $4m^2n^2\sqrt{5n}$.
14. $3a\sqrt{11ab}$.
15. $4mn^2\sqrt{7n}$.
16. $6ax\sqrt{3a}$.
17. $5x^3y\sqrt{5y}$.
18. $7mn^2\sqrt{2m}$.
19. $7p^3q^3\sqrt{3pq}$.
20. $8x^3y^2\sqrt{3xyz}$.
21. $9xy\sqrt{5yz}$.
22. $10m^2n\sqrt{5mn}$.
23. $3a\sqrt[3]{2a}$.
24. $2xy\sqrt[3]{4y}$.
25. $2m\sqrt[3]{5mn^2}$.
26. $4a^2b\sqrt[3]{3c}$.
27. $5mn\sqrt[3]{2mn}$.
28. $6xy\sqrt[3]{2x^2y}$.
29. $2a^2b\sqrt[4]{2bc^3}$.
30. $3ax\sqrt[4]{2ay}$.
31. $5b\sqrt[4]{2ab}$.
32. $2ax\sqrt[5]{2ax^2y}$.
33. $3ax\sqrt[6]{2}$.
34. $(a + b)\sqrt{3}$.
35. $(2a - 3b)\sqrt{3}$.
36. $(x - y)\sqrt{2a}$.

37. $(5a - 4)\sqrt{5b}$.

38. $(3x - 2)\sqrt{3a}$.

39. $(2x + 5y)\sqrt{3}$.

40. $(3a - 7b)\sqrt{2}$.

41. $(3x - 7y)\sqrt{5x}$.

42. $(x^{\frac{1}{2}} + 2y^{\frac{1}{3}})\sqrt{5}$.

43. $(2a^2 - 8)\sqrt{2}$.

44. $(x - 3)\sqrt[3]{4}$.

45. $(x - y)\sqrt[3]{3}$.

Art. 282.

1. $\sqrt{3}$.
2. $\sqrt{5}$.
3. $2\sqrt{2}$.
4. $\sqrt{10}$.
5. $\sqrt[3]{10}$.
6. 2.
7. $\sqrt{3}$.
8. $\sqrt{2}$.
9. $\sqrt{2}$.
10. $\sqrt{2}$.
11. $\sqrt[3]{7}$.
12. $\sqrt{6}$.
13. $\sqrt[3]{4}$.
14. $\sqrt[3]{3}$.
15. $\sqrt{5}$.
16. $\sqrt[3]{6}$.
17. $\sqrt[3]{3a}$.
18. $\sqrt[3]{2ab^2}$.
19. $y\sqrt[3]{3xy}$.
20. $\sqrt[3]{10ab^2}$.
21. $y\sqrt{10xy}$.
22. $\sqrt[4]{10a^2b}$.
23. $b\sqrt{2a}$.
24. $\sqrt[3]{5xy^2}$.
25. $ab\sqrt{15b}$.
26. $\sqrt[3]{30}$.
27. $x\sqrt{5a}$.
28. $m\sqrt{6a}$.
29. $b\sqrt{19a}$.
30. $x\sqrt{10a}$.
31. $bc\sqrt{77ac}$.
32. $a\sqrt{105b}$.

Art. 283.

1. $\frac{1}{4}\sqrt{6}$.
2. $\frac{1}{3}\sqrt{6}$.
3. $\frac{1}{6}\sqrt{30}$.
4. $\frac{1}{6}\sqrt{21}$.
5. $\frac{1}{6}\sqrt{22}$.
6. $\frac{1}{10}\sqrt{38}$.
7. $\frac{1}{7}\sqrt{21}$.
8. $\frac{1}{2}\sqrt[3]{4}$.
9. $\frac{1}{3}\sqrt[3]{21}$.
10. $\frac{1}{4}\sqrt[3]{44}$.
11. $\frac{1}{5}\sqrt[3]{70}$.
12. $\frac{1}{6}\sqrt[3]{105}$
13. $\frac{2}{7}\sqrt[3]{14}$.
14. $\frac{1}{3}\sqrt[3]{15}$.
15. $\frac{1}{2}\sqrt[4]{14}$.
16. $\frac{1}{3}\sqrt[4]{57}$.
17. $\sqrt{2}$.
18. $\sqrt{6}$.
19. $2\sqrt{5}$.
20. $\sqrt[3]{4}$.

21. $\frac{1}{3}\sqrt[3]{6}$.

22. $\frac{1}{4}\sqrt[3]{100}$.

23. $\frac{1}{3}\sqrt[3]{20ab}$.

24. $\frac{x^2}{2a}\sqrt[3]{5a}$.

25. $\frac{3a}{5xy}\sqrt{5bx}$.

26. $\frac{n}{xy^2}\sqrt{10my}$.

27. $\frac{b}{2xy}\sqrt{110ay}$.

28. $\frac{2a}{x^2y^3}\sqrt{13bxy}$.

29. $\frac{a}{2x}\sqrt[3]{180a^2y}$.

30. $ab\sqrt{2}$.

31. $3a^2\sqrt{3}$.

32. $\frac{a}{2b}\sqrt{7ab}$.

33. $\sqrt{x^2-y^2}$.

34. $\frac{1}{2}\sqrt{a(a+x)}$.

35. $(a+x)\sqrt{a(a-x)}$.

36. $\sqrt{5(x^2-y^2)}$.

Art. 285.

1. $\sqrt{19}$.
2. $\sqrt{45}$.
3. $\sqrt{8}$.
4. $\sqrt{320}$.
5. $\sqrt{10}$.
6. $\sqrt[3]{54}$.
7. $\sqrt[3]{95}$.
8. $\sqrt[4]{32}$.
9. $\sqrt[3]{54ab}$.
10. $\sqrt[3]{8a^7b^5}$.
11. $\sqrt{18ab^3}$.
12. $\sqrt[3]{40ax}$.
13. $\sqrt{40m^2n^4}$.
14. $\sqrt[4]{\frac{14a}{9}}$.
15. $\sqrt[4]{3x^5y}$.
16. $\sqrt{x^2-9}$.
17. $\sqrt{\frac{a+b}{a-b}}$.
18. $\sqrt{x^2-9}$.
19. $\sqrt{\frac{x+a}{x-a}}$.

Art. 287.

1. $\sqrt[6]{27}$; $\sqrt[6]{16}$.
2. $\sqrt[6]{125}$; $\sqrt[6]{49}$.
3. $\sqrt[12]{16}$; $\sqrt[12]{27}$.

4. $\sqrt[4]{25}$; $\sqrt[4]{10}$.

5. $\sqrt[6]{8}$; $\sqrt[6]{9}$; $\sqrt[6]{7}$.

6. $\sqrt[12]{64}$; $\sqrt[12]{256}$; $\sqrt[12]{1000}$.

7. $\sqrt[10]{2}$; $\sqrt[10]{243}$; $\sqrt[10]{16}$.

8. $\sqrt[18]{64}$; $\sqrt[18]{125}$; $\sqrt[18]{121}$.

9. $\sqrt[12]{64}$; $\sqrt[12]{256}$; $\sqrt[12]{125}$; $\sqrt[12]{49}$.

10. $\sqrt[20]{1024}$; $\sqrt[20]{625}$; $\sqrt[20]{243}$; $\sqrt[20]{144}$.

11. $\sqrt[24]{a^8}$; $\sqrt[24]{b^6}$; $\sqrt[24]{c^4}$; $\sqrt[24]{a^8x^8}$.

12. $\sqrt[12]{x^6y^6}$; $\sqrt[12]{x^4y^8}$; $\sqrt[12]{a^6b^3}$; $\sqrt[12]{a^2b^4}$.

13. $\sqrt[6]{(a+x)^3}$; $\sqrt[6]{(x-y)^2}$.

14. $\sqrt[12]{(m-n)^3}$; $\sqrt[12]{(m+n)^2}$.

15. $\sqrt[24]{(x+y)^4}$; $\sqrt[24]{(m+1)^8}$.

16. $\sqrt[18]{(x^2-1)^2}$; $\sqrt[18]{(x^2+1)^3}$.

17. $3\sqrt[12]{64}$; $2\sqrt[12]{16}$; $3\sqrt[12]{8}$; $2\sqrt[12]{4}$.

18. $5\sqrt[12]{729}$; $3\sqrt[12]{16}$; $3\sqrt[12]{125}$; $3\sqrt[12]{4}$.

19. $10\sqrt[18]{512}$; $5\sqrt[18]{729}$; $3\sqrt[18]{343}$; $2\sqrt[18]{121}$.

Art. 289.

1. $20\sqrt{2}$.
2. $2\sqrt{3}$.
3. $18\sqrt{3}$.
4. $38\sqrt{6}$.
5. $11\sqrt[3]{5}$.
6. $11\sqrt[3]{2}$.
7. $\frac{2}{3}\sqrt{10}$.
8. $8\sqrt[3]{4}$.
9. $2\sqrt{5}$.
10. $6\sqrt[3]{6}$.
11. $40\sqrt{5}$.
12. $\sqrt{3}$.
13. $22\sqrt{5}$.
14. $4\sqrt{3}$.
15. $6y\sqrt{3x}$.
16. $(3a^2-4a)\sqrt{2a}$.
17. $2\sqrt{2a}$.
18. $8\sqrt{7}-\sqrt[3]{3}$.
19. $(a+x)^2\sqrt[3]{ax}$.
20. $2xy\sqrt[3]{y-z}$.

Art. 291.

1. $5\sqrt{10}$.
2. 36.
3. $48\sqrt{3}$.
4. 8.
5. $15\sqrt[3]{4}$.
6. $8\sqrt[4]{6}$.
7. $15\sqrt[4]{15}$.
8. $a^2b^2\sqrt{bx}$.
9. $6x^2y^2$.
10. $45abx\sqrt[3]{2x}$.
11. $14a^2b^2\sqrt[4]{ab}$.
12. $36x^2y^2\sqrt[4]{2}$.
13. $72a\sqrt[3]{4b^2}$.
14. 225.
15. $5 + 2\sqrt{6}$.
16. $8 - 2\sqrt{15}$.
17. 4.
18. $28 - 7\sqrt{15}$.
19. $12 - 5\sqrt{35}$.
20. $10\sqrt{15} + 4$.
21. $\sqrt[3]{4} + 2\sqrt[3]{6} + \sqrt[3]{9}$.
22. $\sqrt[3]{25} - \sqrt[3]{49}$.
23. $12\sqrt[3]{2} + 10\sqrt[3]{3} - 6\sqrt[3]{36}$.
24. $a^2b - x^2y$.
25. $a^2b\sqrt[3]{b} - x^2y\sqrt[3]{y}$.
26. $2\sqrt{15} - 6$.
27. $5\sqrt{35} + 5\sqrt{21} - \sqrt{15} - 15$.
28. 2.
29. 6.
30. $5 + 2\sqrt[3]{45} - \sqrt[3]{36} - \sqrt[3]{6} - 2\sqrt[3]{10}$.
31. $\sqrt[3]{a^2} - \sqrt[3]{b^2} - \sqrt[3]{c^2} + 2\sqrt[3]{bc}$.
32. $2\sqrt[3]{2} + 2\sqrt[3]{6} - 2\sqrt[3]{4} - 2\sqrt[3]{3} - 2$.
33. $5 - \sqrt[4]{54} + 2\sqrt[4]{3} - \sqrt[4]{24} - \sqrt[4]{72}$.

Art. 293.

1. $\sqrt[4]{108}$.
2. $\sqrt[12]{6912}$.
3. $\sqrt{2ax}$.
4. $6\sqrt[6]{8a^3x^5y^3}$.
5. $3\sqrt[6]{32}$.
6. $12\sqrt{3}$.
7. $\sqrt[6]{72}$.
8. $\sqrt[6]{2000}$.

9. $8\sqrt[6]{972}$.

10. $45\sqrt[4]{3}$.

11. $3\sqrt[6]{2}$.

12. $15\sqrt[6]{135}$.

13. $ab\sqrt[6]{b^3c^2}$.

14. $48\sqrt[3]{3}$.

15. $\sqrt[6]{(a-x)^5}$.

16. $\frac{1}{3}\sqrt[15]{49\times 3^{18}}$.

17. $\frac{1}{10}\sqrt[6]{9000}$.

18. $\frac{1}{3}\sqrt[6]{486}$.

19. $20b\sqrt[6]{a^5b}$.

20. $3x\sqrt[6]{3x}$.

21. $\frac{1}{18}\sqrt[6]{2592}$.

22. $\sqrt[6]{54}$.

23. $3a\sqrt[10]{32a^9x^7}$.

24. $b\sqrt[6]{a^5b}$.

25. $24a\sqrt[6]{2a}$.

26. $60a\sqrt[6]{108a}$.

27. $\frac{1}{6}\sqrt[4]{250}$.

28. $6\sqrt[12]{x^9y^4}$.

29. $30\sqrt{2}$.

30. $4\sqrt[12]{2\times 3^{10}}$.

31. $\sqrt[3]{a^2}-b\sqrt[3]{b}$.

32. $6a\sqrt[10]{288ax^9}$.

33. $\sqrt[3]{4}-5$.

34. $2\sqrt[3]{2}+2\sqrt[6]{432}+3$.

35. $10\sqrt[3]{4}+11\sqrt[6]{108}-18$.

36. $3\sqrt[6]{7^5}+\sqrt[3]{35}-3\sqrt{35}-\sqrt[6]{5^5}$.

37. $\sqrt[6]{32}-\sqrt[3]{6}+\sqrt[6]{500}-\sqrt{6}+\sqrt[6]{243}-\sqrt{15}$.

38. $\sqrt[3]{4}-2$.

39. $\sqrt[10]{128}+\sqrt[5]{6}-\sqrt[5]{4}-\sqrt{6}-\sqrt[10]{2187}+\sqrt[10]{972}$.

40. $\sqrt[6]{a^5}+\sqrt[4]{a^2b}-\sqrt[6]{a^3c}-\sqrt[12]{a^4b}-\sqrt[3]{b}+\sqrt[12]{bc^2}$.

41. $6\sqrt[10]{128}-15\sqrt[5]{6}-2\sqrt[10]{320}+5\sqrt[10]{90}$.

42. $6\sqrt[3]{a^2x}-4\sqrt[12]{a^9x^4}+9\sqrt[6]{a^4y^3}-6\sqrt[4]{a^3y^3}$.

43. $48+20\sqrt[6]{72}-18\sqrt{6}-15\sqrt[6]{243}$.

44. $2 + 2\sqrt[4]{2} - \sqrt[3]{20} - \sqrt[3]{25}$.

45. $2 - \sqrt[3]{9} + \sqrt[6]{1125} + \sqrt{10}$.

46. $2\sqrt[4]{a^3} - 3\sqrt[3]{ab} + 5\sqrt[6]{a^2b^3} - 2\sqrt{ab} + 3\sqrt[6]{b^5} - 5\sqrt{b}$.

47. $-3 - 2\sqrt[6]{72} + \sqrt[3]{9}$.

48. $\sqrt[4]{a^3} + \sqrt[6]{b^5}$.

49. $5 + \sqrt{6}$.

50. $\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}$.

51. $\sqrt{x} - \sqrt{y}$.

52. $\sqrt[3]{125} - \sqrt[3]{27}$.

53. $2x - 6\frac{1}{4}\sqrt[4]{xy^3} + 5\frac{1}{4}y$.

54. $\sqrt[3]{x^2} - 13\sqrt[3]{xy} + 4\sqrt[3]{y^2}$.

Art. 295.

1. $\sqrt{5}$.
2. 6.
3. $4\sqrt{3}$.
4. $5\sqrt{ax}$.
5. $2\sqrt{6}$.
6. $a\sqrt[3]{bc}$.
7. $\sqrt[3]{a+b}$.
8. $\frac{1}{a-x}\sqrt{a^3-x^2}$.
9. $3\sqrt{2}$.
10. $4\frac{1}{2}\sqrt[3]{4}$.
11. $\frac{3}{4}\sqrt{10}$.
12. $\frac{3}{4}\sqrt[4]{40}$.
13. 3.
14. $3\sqrt[3]{2}$.
15. $2\sqrt[3]{2}$.
16. 4.
17. $\frac{1}{2}\sqrt{6}$.
18. $\frac{1}{10}$.

Art. 297.

1. $\sqrt[3]{3}$.
2. $\frac{1}{2}\sqrt[6]{48}$.
3. $\sqrt[12]{8a}$.
4. $\frac{1}{2}\sqrt[6]{480}$.

5. $\frac{1}{2}\sqrt[6]{486}$.
6. $2\sqrt[6]{3}$.
7. $\sqrt[6]{4ab}$.
8. $\frac{2}{3}\sqrt[4]{27}$.
9. $\frac{1}{3}\sqrt[6]{1944}$.
10. $\sqrt[12]{18ax^2}$.
11. $\sqrt[15]{4a^4x^6}$.
12. $5\sqrt[6]{x^5y^5}$.
13. $3\sqrt[15]{16}$.
14. $\frac{1}{2}\sqrt[10]{10368}$.
15. $\frac{1}{3}\sqrt[12]{3^{11}a^2b}$.
16. $5\sqrt[3]{2}$.
17. $6\sqrt[3]{4}$.
18. $\frac{3}{4}\sqrt[3]{7}$.
19. $\sqrt[3]{a} + \sqrt[5]{b^2}$.
20. $\sqrt[3]{x} + \sqrt[4]{y}$.
21. $\sqrt[3]{a} + \sqrt[12]{a^2b^3} + \sqrt{b}$.

Art. 299.

1. $x = 7$.
2. $x = 2$.
3. $x = 13$.
4. $x = 20$.
5. $x = 1$.
6. $x = 16$.
7. $x = 9$.
8. $x = 25$.
9. $x = 81$.
10. $x = 2$.
11. $x = 4$.
12. $x = 4$.
13. $x = \frac{1}{2}$
14. $x = 20$.
15. $x = 2$.
16. $x = 21$.
17. $x = 9$.
18. $x = 4$.
19. $x = 2\frac{1}{2}$.
20. $x = 9$.
21. $x = 25$.
22. $x = \frac{9}{25}$.
23. $x = 1$.
24. $x = 85$.

Art. 301.

1. $x = 4$.
2. $x = \frac{3}{16}$.
3. $x = 10$.
4. $x = -1\frac{1}{2}$.

5. $x = 3$.
6. $x = 64$.
7. $x = \frac{4n}{(1+n)^2}$.
8. $x = \frac{1}{8}\frac{1}{1}$.
9. $x = \frac{1}{3}$.
10. $x = 1\frac{1}{3}$.
11. $x = 6$.
12. $x = 4$.
13. $x = \left(\frac{ab}{a-b}\right)^2$.
14. $x = \frac{9b^2}{a}$.
15. $x = \frac{b(a+b)^2}{(a-b)^2}$.

Art. 310.

1. $x = \pm 4$.
2. $x = \pm 6$.
3. $x = \pm 4$.
4. $x = \pm 7$.
5. $x = \pm \sqrt{5}$.
6. $x = \pm 3$.
7. $x = \pm 7$.
8. $x = \pm 4$.
9. $x = \pm 11$.
10. $x = \pm 3$.
11. $x = \pm 2$.
12. $x = \pm 9$.
13. $x = \pm 2$.
14. $x = \pm 3$.
15. $x = \pm 2\frac{1}{3}$
16. $x = \pm \frac{1}{2}$.
17. $x = \pm 1$.
18. $x = \pm \frac{8}{3}$.
19. $x = \pm 5$.
20. $x = \pm 3$.
21. $x = \pm 5$.
22. $x = \pm 2$.
23. $x = \pm 2$.
24. $x = \pm \frac{5}{8}$.
25. $x = \pm \sqrt{3}$.
26. $x = \pm \frac{1}{2}\sqrt{3}$.
27. $x = \pm \frac{1}{2}\sqrt{3}$.
28. $x = \pm 2a$.
29. $x = \pm 2$.
30. $x = \pm \frac{1}{2}\sqrt{3}$.
31. $x = \pm \frac{1}{2}\sqrt{5}$.
32. $x = \pm \frac{1}{2}$.
33. $x = \pm \frac{1}{2}a\sqrt{3}$.
34. $x = \frac{\pm 2}{\sqrt{4ab - b^2}}$.

35. $x = \pm \sqrt{2ab - b^2}$.

36. $x = \pm \dfrac{2a\sqrt{b}}{1 - b}$

Art. 316.

1. $x = 9$, or -1.
2. $x = 9$, or 3.
3. $x = 2$
4. $x = 3$.
5. $x = 1$, or -17.
6. $x = 19$, or -1.
7. $x = -2$, or -18.
8. $x = 11$.
9. $x = 25$, or -1.
10. $x = 4$, or -2.
11. $x = 7$, or -3.
12. $x = 4$, or 2.
13. $x = 9$, or -1.
14. $x = 4$, or -2.
15. $x = -1$, or -3.
16. $x = 6$, or -2.
17. $x = 1\frac{1}{2}$, or $-\frac{1}{2}$.
18. $x = 2\frac{1}{2}$, or $-\frac{1}{2}$.
19. $x = 1\frac{1}{5}$, or $-\frac{2}{5}$.
20. $x = 4\frac{1}{2}$, or $-1\frac{1}{2}$.
21. $x = 1$, or $-\frac{4}{5}$.
22. $x = 1\frac{1}{2}$, or $-3\frac{1}{2}$.
23. $x = 1\frac{3}{4}$, or $-2\frac{3}{4}$.
24. $x = 7$, or $-6\frac{1}{3}$.
25. $x = \frac{3}{2}$, or $-\frac{5}{6}$.
26. $x = 3$, or -11.
27. $x = 5$, or $-4\frac{1}{4}$.
28. $x = 3$, or $-4\frac{2}{3}$.
29. $x = 1\frac{1}{2}$, or $-\frac{5}{6}$.
30. $x = 2$; or -1.
31. $x = 7$, or $\frac{1}{3}$.
32. $x = 5\frac{2}{5}$, or 5.
33. $x = 10$, or $-\frac{2}{5}$.

Art. 318.

1. $x = 49$, or 4.
2. $x = 11$, or 4.
3. $x = 8$, or $-\frac{8}{3}$.
4. $x = 9$, or -5.
5. $x = \frac{2}{3}$, or -3.
6. $x = 18$, or 3.
7. $x = 4$, or $-14\frac{2}{3}$.
8. $x = 8$, or $-\frac{8}{3}$.
9. $x = 4$, or $7\frac{1}{3}$.
10. $x = 25$, or 3.
11. $x = \frac{3}{5}(7 \pm \sqrt{55})$.

12. $x = 4$, or -21.
13. $x = 4$, or $\frac{1}{5}$.
14. $x = 4$, or -25.
15. $x = 49$, or 25.
16. $x = 4$, or $\frac{49}{11}$.
17. $x = 5$, or $-\frac{1712}{216}$.
18. $x = 3$, or 2.
19. $x = 9$, or $-\frac{1}{5}$.
20. $x = 7$, or $9\frac{1}{4}$.
21. $x = 6$, or $\frac{13}{6}\frac{2}{2}\frac{3}{5}$.
22. $x = 1$, or $\frac{1}{9}$.
23. $x = 8$, or $6\frac{4}{4}\frac{4}{9}$.
24. $x = 16$, or 0.
25. $x = \pm 5$.
26. $x = 4$, or $\frac{1}{4}$.
27. $x = 1$, or $-\frac{2}{3}\frac{5}{5}$.
28. $x = 1$, or $\frac{1}{21}$.
29. $x = 8$, or $-\frac{2}{6}\frac{3}{3}$.
30. $x = \frac{1}{2}$, or $-4\frac{1}{6}$.
31. $x = 0$, or $\frac{1}{2}\frac{6}{5}$.
32. $x = \pm \frac{1}{3}\sqrt{3}$.
33. $x = 4\frac{1}{4}$, or $\frac{1}{4}$.
34. $x = 3$, or $\frac{1}{3}$.
35. $x = a$, or $\frac{a}{5}$.
36. $x = 4a$, or $3a$.
37. $x = a(-1 \pm \sqrt{2})$.
38. $x = 7$, or $\frac{7}{25}$.
39. $x = 1$, or $\frac{1}{81}$.
40. $x = \pm 2a$.

Art. 321.

1. $x = 3$, or 2.
2. $x = 5$, or -2.
3. $x = \frac{1}{3}$, or 7.
4. $x = \frac{8}{3}$, or $-1\frac{2}{6}$.
5. $x = 1\frac{1}{4}$, or $-2\frac{1}{3}$.
6. $x = 2$, or $-1\frac{1}{2}$.
7. $x = -1\frac{2}{3}$, or $-\frac{1}{3}$.
8. $x = 3\frac{1}{2}$, or $-3\frac{1}{4}$.
9. $x = 5$, or $-5\frac{2}{3}$.
10. $x = 10$, or $-11\frac{1}{3}$.
11. $x = 3$, or $1\frac{2}{3}$.
12. $x = 3$, or $-4\frac{1}{5}$.
13. $x = 6$, or $-4\frac{1}{2}$.
14. $x = 5$, or $-4\frac{1}{3}$.
15. $x = 4$, or $-1\frac{1}{7}$.
16. $x = 7$, or $-7\frac{1}{3}$.
17. $x = 5$, or $-6\frac{1}{3}$.
18. $x = 3$, or $6\frac{1}{11}$.
19. $x = 9$, or $-11\frac{2}{3}$.
20. $x = 10$, or $-2\frac{1}{4}$.

21. $x = 25$, or 1.

22. $x = 5\frac{1}{2}$, or $-2\frac{1}{7}$.

23. $x = 9\,(1 \pm 2\sqrt{-1})$.

24. $x = -1\frac{1}{3}$, or $-5\frac{1}{3}$.

25. $x = 10\frac{1}{2}$, or $-1\frac{1}{2}$.

26. $x = 7$, or $-11\frac{2}{3}$.

27. $x = 152$, or 76.

Art. 323.

1. $x = \dfrac{b \pm \sqrt{b^2 + 4ac}}{2a}$.

2. $x = \dfrac{-a \pm \sqrt{a^2+bc}}{c}$.

3. $x = 2a$, or $\dfrac{a}{2}$.

4. $x = \dfrac{9a \pm a\sqrt{-279}}{20}$.

5. $x = \dfrac{b}{a+b}$,

or $-\dfrac{a}{a+b}$.

6. $x = \dfrac{3a}{4}$, or $\dfrac{a}{2}$.

7. $x = \dfrac{-11a \pm a\sqrt{13}}{6}$.

8. $x = \dfrac{c \pm \sqrt{c^2 + 4ac}}{2\,(a+b)}$.

9. $x = \dfrac{b \pm \sqrt{b^2+4c(a-c)}}{2\,(a-c)}$.

10. $x = \dfrac{2c + ad}{d\,(a+b)}$,

or $\dfrac{c}{d\,(a-b)}$.

11. $x = \dfrac{2a-b}{ac}$,

or $-\dfrac{3a+2b}{bc}$.

12. $x = \dfrac{a^2 \pm 1}{a}$.

13. $x = \dfrac{a+b}{a-b}$,

or $\dfrac{a-b}{a+b}$.

Art. 325.

1. -3, or -4.

2. 3, or 2.

3. -6, or 3.

4. 12, or -7.

5. 11, or -1.

6. 25, or -4.

7. -17, or 4.

8. -100, or 4.

9. 15, or 13.

10. -45, or 25.

11. $3a$, or $-2a$.

12. $\frac{a}{b}$, or $-\frac{b}{a}$.

13. $1\frac{1}{3}$, or $-\frac{3}{5}$.

14. $\frac{ac}{b}$, or $-\frac{bc}{a}$.

15. $\frac{1}{4}$, or $\frac{3}{5}$.

16. 10, or -29.

17. $\frac{a^2+1}{a}$, or $\frac{a^2-1}{a}$.

18. $\frac{5c}{3}$, or $-\frac{29c}{21}$.

19. $a+b$, or $a-b$.

20. $\frac{mn}{m+n}$, or $-\frac{m+n}{m-n}$.

Art. 327.

1. $1\frac{1}{2}$, or $1\frac{3}{5}$.

2. 2, or $-2\frac{1}{3}$.

3. $1\frac{1}{3}$, or $-1\frac{3}{5}$.

4. $-2\frac{1}{3}$, or $-2\frac{1}{7}$.

5. 4, or $-3\frac{3}{5}$.

6. $12\frac{1}{2}$, or $-13\frac{1}{3}$.

7. $\frac{7}{3}$, or $-1\frac{1}{2}$.

8. $1\frac{1}{7}$, or $-1\frac{3}{5}$.

9. $-\frac{3}{5}$, or $-\frac{5}{8}$.

10. $1\frac{1}{3}$, or $-3\frac{1}{3}$.

11. $1\frac{3}{5}a$, or $-1\frac{1}{2}b$.

12. $\frac{m}{n}$, or $-\frac{n}{m}$.

13. $\frac{3}{5}$, or $-1\frac{1}{3}$.

14. 3, or $-1\frac{11}{13}$.

15. -1, or $\frac{3}{5}$.

16. $\frac{3}{5}$, or $1\frac{3}{5}$.

17. 2, or $-1\frac{3}{4}$.

18. $\frac{4}{15}$, or $-4\frac{3}{5}\frac{5}{8}$.

19. $\frac{c}{a}$, or $-\frac{d}{b}$.

20. $\frac{b}{2a}$, or $\frac{d}{c}$.

21. $\frac{m-n}{mn}$, or $-\frac{m+n}{mn}$.

22. $\frac{bc}{a}$, or $-\frac{1}{c}$.

Art. 328.

1. ± 10, ± 15.

2. 30 yrs.; 25 yrs.

3. 40 ft.; 100 ft.

4. $24\frac{3}{4}$ yds. at $\$6\frac{3}{5}$.

5. 6; 21.

6. 14; 21; 35.

7. 40 weeks at $\$35$.

8. ± 2.

9. 224.
10. 60 apples; 20 pears.
11. 4550.
12. $150.
13. 30 acres.
14. 12 rods.
15. 8 and 9.

Art. 329.

1. 40 and 16.
2. 7 and 1.
3. 120.
4. 16.
5. 11.
6. 7.
7. 27 lb.
8. 60 yds. at 75 cts.
9. 32 rods by 24 rods.
10. $250.
11. A, 24; B, 20.
12. 40.
13. $40, or $60.
14. 84.
15. 24 yds.
16. $.08; $.10.
17. A, 45 da.; B, 36 da.
18. $.20.
19. 96 rods; 126 rods.
20. $240.
21. 6 hrs.; 12 hrs.
22. 25; 14.
23. 5 miles; 3 miles.
24. 15 days; 30 days.
25. 44 yds.; 110 yds.
26. 4 miles per hour.
27. 50 gals.; 60 gals.
28. 12 miles, or 8 miles.
29. 5 miles per hour.
30. 36 rods; 40 rods.
31. 45 chickens;
64 turkeys.

Art. 333.

1. $x = 9$, or $\sqrt[3]{1681}$.
2. $x = \pm 8$,
or $\pm \sqrt{-11^3}$.
3. $x = 729$, or $(-\frac{1}{4}^3)^6$.
4. $x = 3$, or $-\sqrt[3]{41}$.
5. $x = 4$, or $\frac{1}{2}\sqrt[3]{2}$.
6. $x = 1$, or $\frac{1}{3}\sqrt[3]{63}$.
7. $x = 1$, or -2.
8. $x = 1$, or $\frac{1}{3}$.

9. $x = \pm\sqrt{-6}$, or $\pm\sqrt{2}$.
10. $x = 16$, or $(\frac{1}{9}\frac{1}{6})^4$.
11. $x = 4$, or $\sqrt{49}$.
12. $x = 243$, or $\sqrt[5]{-28^5}$.
13. $x = 125$, or -8.
14. $x = \pm 8$.
15. $x = 32$, or $\pm\frac{1}{3}\sqrt{-57^5}$.
16. $x = 1, 1$, or $1 \pm 2\sqrt{15}$.
17. $x = 9$, or -12.
18. $x = 4$, or 69.
19. $x = 2, 1\frac{1}{2}$, or $\frac{1}{4}(7 \pm \sqrt{33})$.
20. $x = 5, -1$, or $2 \pm \sqrt{-14}$.
21. $x = 3, \frac{1}{3}$, or $\frac{1}{3}(-8 \pm \sqrt{55}$
22. $x = -12$, or -5.
23. $x = 5$, or $37\frac{1}{2}$.
24. $x = 10$, or -2.
25. $x = 4$, or -1.

Art. 340.

1. $x = 3$, or $3\frac{1}{3}\frac{1}{5}$; $y = 2$, or $3\frac{4}{7}$.
2. $x = 3$, or $-\frac{5}{2}\frac{3}{7}$; $y = -4$, or $\frac{2}{2}\frac{2}{7}\frac{7}{7}$.
3. $x = 2$, or -46; $y = 3$, or 15.
4. $x = 7$, or 5; $y = -5$, or -7.
5. $x = 8$, or $17\frac{2}{3}$; $y = 6$, or $-13\frac{1}{3}$.
6. $x = 2$, or $-\frac{1}{3}$; $y = 4$, or $1\frac{2}{3}$.
7. $x = 5$, or $-9\frac{1}{4}$; $y = 3$, or $-6\frac{1}{2}$.
8. $x = 5$, or $6\frac{7}{1}\frac{7}{9}$; $y = 3$, or $-1\frac{2}{1}\frac{2}{9}$.
9. $x = 2$, or 5; $y = 6$, or 3.
10. $x = \frac{1}{2}$, or $\frac{3}{4}$; $y = \frac{3}{4}$, or $\frac{1}{2}$.
11. $x = 9$, or $-14\frac{1}{1}\frac{1}{6}$; $y = 4$, or $-6\frac{1}{4}$.
12. $x = 5$, $y = 3$.

13. $x = 1$, or -50; $y = 2$, or 19.

14. $x = 4$, or -3; $y = 2$, or $-2\frac{2}{3}$.

15. $x = 4$, or $-\frac{1}{4}$; $y = 1$, or $-3\frac{1}{4}$.

16. $x = 5$, or $-3\frac{4}{7}$; $y = 2$, or $7\frac{19}{21}$.

17. $x = 2$, or $5\frac{1}{2}$; $y = 1$, or $\frac{4}{11}$.

18. $x = 13$, or 67; $x = 7$, or -47.

19. $x = 4$, or $5\frac{1}{3}$; $y = 3$, or $1\frac{2}{3}$.

20. $x = \pm \frac{1}{2}\sqrt{10}$; $y = 2 \mp \frac{1}{2}\sqrt{10}$.

Art. 342.

1. $x = \pm 2$, or $\pm 1\frac{3}{16}\sqrt{15}$; $y = \pm 1$, or $\pm \frac{7}{8}\sqrt{15}$.

2. $x = \pm 3$, or $\pm \frac{4}{3}\sqrt{6}$; $y = \pm 1$, or $\pm \frac{1}{6}\sqrt{6}$.

3. $x = \pm 2$, or ± 2; $y = \pm 3$, or ± 1.

4. $x = \pm 4$, or $\pm \dfrac{5}{\sqrt{2}}$; $y = \mp 3$, or $\pm \dfrac{5}{\sqrt{2}}$.

5. $x = \pm 10$, or $\mp \frac{14}{47}\sqrt{-47}$;
   $y = \pm 3$, or $\pm \frac{57}{47}\sqrt{-47}$.

6. $x = \pm 5$; $y = \pm 2$.

7. $x = \mp 3$, or $\mp 2\frac{4}{5}$; $y = \pm 1$, or $\pm 1\frac{1}{3}$.

8. $x = \pm 3$, or $\pm \frac{5}{2}\sqrt{2}$; $y = \pm 2$, or $\pm \frac{1}{2}\sqrt{2}$.

9. $x = \pm 6$, or ∓ 102; $y = \pm 5$, or ∓ 59.

10. $x = \pm 1$, or $\pm 1\frac{3}{21}\sqrt{21}$; $y = \pm 3$, or $\pm \frac{8}{21}\sqrt{21}$.

11. $x = \pm 3$, or $\pm \dfrac{37}{\sqrt{211}}$; $y = \pm 2$, or $\pm \dfrac{60}{\sqrt{211}}$.

12. $x = \pm 6$, or $\mp \frac{1}{2}\sqrt{2}$; $y = \pm 5$, or $\pm \frac{11}{2}\sqrt{2}$.

13. $x = \pm 2$, or $\mp \frac{3}{7}\sqrt{7}$; $y = \pm 1$, or $\pm \frac{5}{7}\sqrt{7}$.

14. $x = \pm 1$, or $\pm \frac{1}{2}\sqrt{2}$; $y = \pm 2$, or $\pm \frac{3}{2}\sqrt{2}$.

15. $x = \pm 6$, or $\pm 4\sqrt{-1}$, $y = \pm 4$, or $\pm 6\sqrt{-1}$.

16. $x = \pm 7$, or $\pm \frac{11}{2}\sqrt{2}$; $y = \pm 4$, or $\pm \frac{3}{2}\sqrt{2}$.

17. $x = \pm 4$, or $\mp \sqrt{7}$; $y = \pm 2$, or $\pm \sqrt{7}$.

18. $x = \pm 8$, or $\pm \frac{7}{2}\sqrt{-2}$; $y = \pm 3$, or $\pm 5\sqrt{-2}$.

19. $x = \pm 1$, or $\pm \frac{3}{5}$; $y = \pm 2$, or $\pm 2\frac{1}{5}$.

20. $x = \pm \frac{4}{3}$, or $\mp \frac{11}{13}\sqrt{43}$; $y = \pm 1$, or $\pm \frac{15}{13}\sqrt{43}$.

Art. 344.

1. $x = 11$, or 9; $y = 9$, or 11

2. $x = 3$, or 5; $y = 5$, or 3.

3. $x = 5$, or -3; $y = 3$, or -5.

4. $x = 16$, or 4; $y = 4$, or 16.

5. $x = \frac{1}{2}$, or -1; $y = -1$, or $\frac{1}{2}$.

6. $x = 3$, or 2; $y = 2$, or 3.

7. $x = 4$, or -3; $y = 3$, or -4.

8. $x = \pm 6$, or ± 3; $y = \pm 3$, or ± 6.

9. $x = \pm \frac{5}{7}\sqrt{14}$; $y = \pm \frac{1}{7}\sqrt{14}$.

10. $x = 4, 3$, or $-4 \pm \sqrt{-11}$;
$y = 3, 4$, or $-4 \mp \sqrt{-11}$.

11. $x = 8$, or 2; $y = 2$, or 8.

12. $x = \pm 4$, or ± 3.
$y = \pm 3$, or ± 4.

13. $x = \pm 3$, or ± 2; $y = \pm 2$, or ± 3.

14. $x = 1$, or 4; $y = 4$, or 1.

15. $x = \pm 3\sqrt{2}$; $y = \pm \sqrt{2}$.

16. $x = 8$, or 64; $y = 64$, or 8.

17. $x = 9$, or 4; $y = 4$, or 9.

18. $x = 3$, 2, or $\pm \frac{1}{2}\sqrt{10} - 2$;
$y = 2$, 3, or $\mp \frac{1}{2}\sqrt{10} - 2$.

19. $x = 2$, or 3; $y = 3$, or 2.

20. $x = 4$, 2, or $\frac{1}{6}(-13 \pm \sqrt{377})$;
$y = 2$, 4, or $\frac{1}{6}(-13 \mp \sqrt{377})$.

Art. 345.

1. $x = \pm 2$; $y = \pm \frac{1}{2}$.

2. $x = 18\frac{1}{2}$; $y = 15\frac{1}{2}$.

3. $x = 10$, $y = 6$.

4. $x = 38$, or 36; $y = 36$, or 38.

5. $x = 25$; $y = 21$.

6. $x = \frac{1}{3}$, or $\frac{1}{4}$; $y = \frac{1}{4}$, or $\frac{1}{3}$.

7. $x = 10$, or 2; $y = 2$, or 10.

8. $x = 10$, or -9; $y = 9$, or -10.

9. $x = \pm 2$, or $\pm \sqrt{2}$; $y = \pm 4$, or $\pm 3\sqrt{2}$.

10. $x = \pm 3$; $y = \pm 4$.

11. $x = \pm 4$, or ± 3; $y = \pm 3$, or ± 4.

12. $x = \pm 2$, or $\pm \frac{1}{2}(\pm \sqrt{-11} \pm 1)$;
$y = \pm 1$, or $\frac{1}{2}(\pm \sqrt{-11} \mp 1)$.

13. $x = \pm 3$, or ± 5; $y = \pm 36$, or $\mp 11\frac{1}{4}$.

14. $x = 3$, or 6; $y = -\frac{1}{3}$, or $-\frac{2}{3}$.

15. $x = 5$, or 4; $y = -4$, or -5.

16. $x = 6$, or -12; $y = 2$, or -4.

17. $x = 3$, or -6; $y = 2$, or 6.

18. $x = 8$, or $-\frac{64}{13}$; $y = 6$, or $-\frac{18}{13}$.

19. $x = 2$, or 4; $y = -\frac{22}{21}$, or $-8\frac{4}{7}$.

20. $x = \frac{1}{3}$, or $\frac{4}{15}$; $y = \frac{1}{3}$, or $-\frac{2}{7}$.

Art. 346.

1. 9 and 11.
2. 40 rods; 12 rods.
3. 36 rods; 20 rods.
4. 23, 12.
5. 60, and 48 rods.
6. 9 and 5.
7. 7 feet; 5 feet.
8. A, 4 miles;
B, 3 miles.
9. Fore wheels, $13\frac{1}{3}$ ft.;
hind wheels, 15 ft.
10. 150 miles.
11. 45 rods; 35 rods.
12. 36.
13. 28 acres at \$15 per A.
14. 4 and 5.
15. \$.50, mace;
\$.25, cloves.
16. One, \$900;
other, \$2,025.
17. 5; 3.
18. 5 and 3.
19. 5; 4.
20. 5 and 7.
21. 60 acres; 80 acres.
22. 68 sheep, at \$2 each.
23. \$1000; \$11000; 11%
24. \$75; \$10.
25. 5%; 4%.

Art. 357.

1. $2:3$; $3:5$; $3:4$; $3:2$; $2:3$.
2. $a:b$; $x+3:x$; $x-y:x+y$; $x^2-y^2:x^2+y^2$.
3. $\frac{2a}{b}$; $\frac{5}{6}$; $\frac{5}{4}$.
4. $\frac{a-x}{a^2+x^2}$; $2bcy$.
5. 1; $x-3:x+3$.
6. $4:7$.
7. $243:128$.
8. 6 and 8.
9. $\sqrt{25}:\sqrt{64}$; $15:20$; $20:25$; $\sqrt[3]{125}:\sqrt[3]{216}$.
10. 6 and 10.
11. The former.
12. $10:1$.
13. $3:2$.
14. 60; 40.
15. $b:a$.
16. $x-2:x+2$.

Art. 379.

5. 40; 20.
6. $\frac{b^2-a^2}{4a-b}$.
7. $x=\pm 20$, or ± 16; $y=\pm 16$, or ± 20.
8. $2\frac{3}{16}$.
9. 45.
10. $20\frac{6}{8}:10 :: 33:16$.
    $22:10\frac{8}{3} :: 33:16$.
    $22:10 :: 35\frac{1}{8}:16$.
    $22:10 :: 33:15$.
11. $x=\pm 15$; $y=\pm 6$.
12. 32; 24.
13. 25 rds.; 15 rds.
14. ± 15; ± 9.
15. A's, \$300; B's, \$350.
16. $x=\frac{2mn}{n^2+1}$.
17. $1\frac{1}{13}$.
18. 248.

Art. 388.

1. $l=32$, $S=187$.
2. $n=40$, $S=4220$.
3. $a=100$, $d=-3$.
4. 500500.
5. $l=15$, $d=\frac{7}{11}$.
6. 8, 13, 18, 23, 28, 33.

7. $\frac{1}{3}$, $\frac{4}{15}$, $\frac{11}{12}$.
8. — 36057.
9. — 4; 20.
10. A 215 mi. in advance of B.
11. $(2t - 1)\,a$.
12. at^2.
13. Last payment, \$103; amount, \$2704.
14. 4.
15. $17\frac{1}{2}$ in.
16. 11 mi. 3 fur. 32 rd. 4 yd.
17. 1, 3, 5, 7.
18. 3, 5, 7, 9, 11.
19. 16, 20, 24.
20. 30, 40, 50, 60.
21. 1, 3, 5, 7, 9.
22. 234.
23. \$250, \$750, \$1250, \$1750, \$2250, \$2750.
24. 57900 ft.

Art. 397.

1. 1875.
2. $\frac{1}{3}$.
3. 486.
4. $\frac{1}{128}$.
5. 12288.
6. $\frac{41}{1024}$.
7. 765.
8. $5\frac{717}{729}$.
9. 15, 45, 135, 405, 1215.
10. 2.
11. 2.
12. $16\frac{1}{3}$.
13. 4, 24, 144, 864.
14. 2, 4, 8.
15. 4, 12, 36.
16. 150 feet.
17. \$120; \$60; \$30.

ANSWERS TO PART II.

Art. 398.

1. 4.
2. 14.
3. 6.
4. 4.
5. 23.
6. $8\frac{7}{16}$.
7. 5.
8. 0.
9. 4.
10. $2\frac{9}{100}$.
11. 80.
12. 0.
13. 8.
14. $\frac{4}{5}$.
15. 1.
16. $16\frac{1}{3}$.
17. 64.
18. 4.
19. 12.
20. 6.
21. 57.
22. 2.
23. 8.
24. 8.
25. 3.
26. 16.
27. 8.
28. 2.
29. 15.

Art. 399.

1. $30a - 3b - 26c + 4f + g$.
2. $14x^2 - x^3 - 17b$.
3. $-\sqrt{m+n}$.
4. $x^{\frac{1}{2}} + 6y^2 + 18$.
5. $ab + 8b^2 - b^4$.
6. $b^2x + 3c^2$.
7. $4a^m + 2x^p - x^q$.
8. $2a^{-1} - b^{-2} + 20c^2$.
9. $7\sqrt{x} - x + 10$.
10. $6a^2c^{\frac{1}{3}} - 4c^2a^{\frac{1}{3}} + 4a^{\frac{1}{3}}c^{\frac{1}{3}}$.
11. $10(a - x^2) - 5\sqrt{a - x}$.
12. $15a^m + 33b^n - 29c^p + 37d^q$.
13. $4a^3 - \frac{1}{2}a^2x + \frac{7}{6}ax^2 + \frac{1}{12}x^3$.
14. $\frac{17}{2}a - \frac{347}{600}b + \frac{25}{8}c - \frac{71}{20}d$.
15. $-3a^2x^2 + 9axy - 7aby^2$.
16. $3\sqrt{x^2 - y^2} + (x + y) - 6\sqrt{a + x}$.
17. $a^2 + 4ax - x^2 + 4a^2x + 8ax^2 + 2ax^3 + 4a^2x^2$.
18. $14axy - 9abc + 3x\sqrt{x + y}$.
19. $17ab^3x - 7bcx^2 - 7$.
20. $2a^4 + 8a^3b^2 + 4cd^2 + 7d + 3a^2$.

Art. 400.

1. $12x^4 - 192$.
2. $a^3 + b^3 - c^3 + 3abc$.
3. $x^7 + 6x^3 + 8x^2 - 1$.
4. $x^3 - 28x^2 + 48x - 21$.
5. $12x^5 - 13x^4 - 34x^3 + 35x^2$.
6. $6x^4 - 96$.
7. $32a^4 + 54ab^3 - 81b^4$.
8. $x^6 - 2x^3 + 1$.
9. $6x^{-6}y^m - 10x^{m-4}y^{m-n} + 2x^{-4}y^m z^{-1}$.
10. $\frac{1}{3} - 6x^2 + 27x^4$.
11. $-2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8$.
12. $\frac{7}{4}x^5 - 4x^4 + \frac{7}{8}\frac{1}{1}x^3 - \frac{4}{4}\frac{3}{3}x^2 - \frac{3}{4}\frac{3}{3}x + 27$.
13. $x^{\frac{3}{5}} - y^{\frac{5}{4}}$.
14. $x^4 - y^{-3}$.
15. $\frac{1}{6}a^7b^2 - \frac{4}{4}\frac{7}{0}a^6b^3 + \frac{2}{2}a^5b^4 - 12a^4b^5$.
16. $-\frac{6}{5}x^2 + \frac{1}{3}\frac{1}{5}xy - \frac{1}{3}\frac{0}{5}xz + \frac{1}{4}\frac{5}{5}y^2 + 25yz$.
17. $x^{3n} - 3x^{2n}y^n + 3x^ny^{2n} - y^{3n}$.
18. $\frac{1}{3}a^4 - 1\frac{1}{2}a^3 + \frac{4}{8}\frac{1}{1}a^2 - \frac{2}{4}\frac{3}{3}a + 6$.
19. $\frac{3}{4}x^5 - 4x^4 + \frac{7}{8}\frac{1}{1}x^3 - \frac{4}{4}\frac{3}{3}x^2 - \frac{3}{4}\frac{3}{3}x + 27$.
20. $6x^7 + \frac{3}{1}\frac{9}{6}x^2y^5 - \frac{3}{3}\frac{3}{2}xy^6 + \frac{2}{6}\frac{1}{4}y^7$.

Art. 401.

1. $a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4$.
2. $8m^8 + 7m^7 + 6m^6 + 5m^5 + 4m^4 + 3m^3 + 2m^2 + m$.
3. $1 - 6a + 9a^2$.
4. $\frac{2}{3}a^2 - \frac{1}{2}ab + b^2$.
5. $x + \frac{1}{2}$.
6. $x^2 - \frac{1}{2}ax + a^2$.
7. $a - a^{\frac{1}{2}}b^{\frac{1}{2}} + b$.
8. $x^{\frac{1}{2}} + a^{\frac{1}{2}}x^{\frac{3}{5}} - a^{\frac{3}{5}}x^{\frac{1}{5}} - a^2$.
9. $x^n - y^n$.
10. $x^{2n} - x^ny^n + y^{2n}$.
11. $8a^{\frac{3}{4}}x^{\frac{3}{5}} - 4a^{\frac{1}{4}}xy^{\frac{1}{4}} + 2a^{\frac{3}{4}}x^{\frac{1}{5}}y^{\frac{1}{4}} - y^{\frac{3}{4}}$.
12. $m^3 - m^2n - \frac{1}{3}mn^2 - \frac{2}{2}\frac{2}{7}n^3$.
13. $8a^{\frac{3}{1}\frac{5}{0}} + 4a^{\frac{2}{1}\frac{5}{0}} + \frac{4}{3}a^{\frac{2}{1}\frac{3}{0}} + \frac{1}{3}a^{\frac{1}{1}\frac{1}{0}} + \frac{1}{1}\frac{1}{8}a^{\frac{1}{1}\frac{1}{0}} + \frac{1}{7}\frac{1}{2}a^{\frac{1}{1}\frac{1}{0}} + \frac{1}{4}\frac{1}{3}\frac{1}{2}$.
14. $x^{\frac{3}{2}} + 2x^{\frac{5}{4}} - 4x - 8x^{\frac{3}{4}} - 16x^{\frac{1}{2}} - 32x^{\frac{1}{4}} - 64$.

15. $6a^9 + 6a^{\frac{15}{2}}b^{-\frac{3}{7}} + 7a^6b^{-\frac{4}{7}} + 8a^{\frac{3}{2}}b^{-\frac{6}{7}} + \frac{5}{6}a^3b^{-\frac{5}{7}} + \frac{2}{3}a^{\frac{1}{2}}b^{-\frac{10}{7}} + \frac{4}{3}a^{\frac{3}{2}}b^{-\frac{12}{7}}$.

16. $3m^{\frac{3r}{5}} + \frac{3}{2}m^{\frac{2r}{5}}n^{\frac{2s}{15}} - \frac{1}{2}m^{\frac{r}{5}}n^{\frac{4s}{15}} + \frac{1}{2}n^{\frac{2s}{5}}$.

17. $6x^2 + 3x^{\frac{5}{3}} + \frac{3}{2}x^{\frac{4}{3}} + \frac{3}{4}x + \frac{1}{8}x^{\frac{2}{3}} + \frac{1}{8}x^{\frac{1}{3}} + \frac{1}{8}$.

Art. 403.

1. $(a^2 + b)(a^4 - a^2b + b^2)$.
$(x^2 - y^3)(x^4 + x^2y^3 + y^6)$.
$(x^3 + y^2)(x^6 - x^3y^2 + y^4)$.
$(x^4 - y^5)(x^8 + x^4y^5 + y^{10})$.
$(p^3 - q^4)(p^6 + p^3q^4 + q^8)$.

2. $(x^2 + y^2)(x^8 - x^6y^2 + x^4y^4 - x^2y^6 + y^8)$.
$(x^4 + y^4)(x^8 - x^4y^4 + y^8)$
$(a - b^2)(a^4 + a^3b^2 + a^2b^4 + ab^6 + b^8)$.
$(m + n^2)(m^4 - m^3n^2 + m^2n^4 - mn^6 + n^8)$.
$(a^6 + x^2)(a^{12} - a^6x^2 + x^4)$.

3. $(x^2 + y^2)(x^4 - x^2y^2 + y^4)(x^{12} - x^6y^6 + y^{12})$.
$(a - n)(a^4 + a^3n + a^2n^2 + an^3 + n^4)(a^{10} + a^5n^5 + n^{10})$.
$(p^4 - q^3)(p^{16} + p^{12}q^3 + p^8q^6 + p^4q^9 + q^{12})$.
$(a^6 - y^5)(a^{12} + a^6y^5 + y^{10})$.
$(x^6 + y^4)(x^{12} - x^6y^4 + y^8)$.

4. $(2x^2 + 3y^2)(4x^4 - 6x^2y^2 + 9y^4)$.
$(5x^4 - 4y^5)(25x^8 + 20x^4y^5 + 16y^{10})$.
$(6a^3 - 10b^2)(36a^6 + 60a^3b^2 + 100b^4)$.
$(2m^5 + 4n^3)(4m^{10} - 8m^5n^3 + 16n^6)$.

5. $(\frac{2}{5}x^2 + \frac{3}{4}y^3)(\frac{4}{25}x^4 - \frac{3}{10}x^2y^3 + \frac{9}{16}y^6)$.
$(\frac{2}{3}m^{\frac{1}{2}} - \frac{1}{6}n^{\frac{1}{2}})(\frac{1}{9}m^{\frac{2}{3}} + \frac{1}{6}m^{\frac{1}{3}}n^{\frac{1}{2}} + \frac{2}{36}n)$.
$(\frac{1}{7}x^{\frac{2}{5}} + \frac{3}{35}y^{\frac{2}{7}})(\frac{4}{49}x^{\frac{4}{5}} - \frac{7}{24}x^{\frac{2}{5}}y^{\frac{2}{7}} + \frac{9}{64}y^{\frac{4}{7}})$.

6. $(x^2y^3 + 2)(x^4y^6 - 2x^2y^3 + 4)$.
$(2x^2 - y^3)(16x^8 + 8x^6y^3 + 4x^4y^6 + 2x^2y^9 + y^{12})$.
$(a^2b^5 + x^4y^3)(a^4b^{10} - a^2b^5x^4y^3 + x^8y^6)$.
$(4x^2 + 9y^2)(16x^4 - 36x^2y^2 + 81y^4)$.

Art. 404.

1. $x^2 - 4x - 21$.
2. $x + 7$.
3. $x - 1$.
4. $x - 7$.
5. $x - 3$.
6. $2x + 5$.
7. $x^2 - 1$.
8. $x - 3$.
9. $x - 1$.
10. $x + 1$.
11. $2x - 5$.
12. $x + 3y$.
13. $x^4 - 7x^2y^2 + y^4$.
14. $a^2 + 4ac + 7c^2$.
15. $x^3 + 3x^2 + 3x + 1$.
16. $x^3 + 3x^2 + 3x + 1$.

Art. 405.

1. $x^4 - x^3 - 67x^2 + 25x + 1050$.
2. $42(x^4 + x^3y - xy^3 - y^4)$.
3. $66ab(a^4 - b^4)$.
4. $2x^3 + x^2 - 5x + 2$.
5. $x^4 - 15x^2 - 10x + 24$.
6. $16x^4 - 36x^3 + 30x^2 - 14x - 6$.
7. $x^4 + 4x^3 - 7x^2 - 34x - 24$.
8. $x^4 - 14x^3 + 71x^2 - 154x + 120$.
9. $x^7 + x^6 - 7x^5 - 4x^4 + 15x^3 - 2x^2 - 12x + 8$.
10. $x^5 - 8x^4 + 50x^2 - x - 42$.
11. $x^4 - 10x^3 + 35x^2 - 50x + 24$.
12. $x^4 - 14x^3 + 71x^2 - 154x + 120$.
13. $x^5 - 6x^4 - 2x^3 + 36x^2 + x - 30$.
14. $x^4 - 5x^3 - 23x^2 + 45x + 126$.

Art. 406.

1. $\frac{axy}{a+y}$. 2. $\frac{a+y}{a}$.

3. $\frac{a}{3(a+2x)}$.

4. $\frac{3xy-5y^2}{4x-7y}$.

5. $\frac{x^2+b^2}{x^3}$.

6. $\frac{a^2}{a+2}$.

7. $\frac{c}{2df}$.

8. $\frac{a+b}{a-b}$.

9. $\frac{x^3+4x+4}{x^2+x+1}$.

10. $\frac{3ax-7a}{7x^2-3x}$.

11. $\frac{2x-1}{x-1}$.

12. $\frac{x^2-ax+a^2}{x^3-a^3}$.

13. $\frac{5x^2+1}{9x^3-4x}$.

14. $\frac{x^2-ax+b^2}{x^2+ax-b^2}$.

15. $\frac{a^4+a^3b+a^2b^2-3ab^3}{a^3-b^3}$.

16. $\frac{x-a}{x^2+a}$.

17. $\frac{x+4}{(x-1)^2}$.

18. $\frac{x+y}{x^2-y}$.

19. $\frac{x^2-5n^2}{x^2+5n^2}$.

20. $\frac{ab-a-b+1}{ab+a+b+1}$.

Art. 407.

1. $\frac{x+26}{6}$.

2. $\frac{16x^2-17xy}{18}$.

3. $\frac{5a^2b+4a^3+3b^2}{5b}$.

4. $\frac{169x+77}{105x}$.

5. $\frac{4x^2+3}{3x}$.

6. $\frac{6ax}{a-x}$.

7. $\frac{2x^2-y^2}{x^2-xy}$.

8. $\frac{4a}{a+x}$.

9. $\frac{3x^2}{x^2-1}$.

10. $\frac{4a^3}{a^4-x^4}$.

11. $a-b$.

12. $\frac{13x+12}{x+1}$.

13. $\frac{1}{1-y^4}$.

14. $\frac{b^2+ab}{b^2-x^2}$.

15. $\frac{2x^2+21x+13}{(1-x^2)(1-4x^2)}$.

16. $\frac{2}{x^4-y^4}$.

17. 0.

18. $\frac{1}{x+2}$.

19. 0.

20. 1.

21. $\frac{(x-4)(x-7)}{x^2}$.

22. $\frac{x+2}{x}$.

23. $\frac{3x-1}{x-2}$.

24. $\frac{a^3-a^2b}{x}$.

25. $\frac{4ab}{a^2-b^2}$.

26. $\frac{x^2}{a^2-x^2}$.

27. $\frac{1}{(m-n)^2}$.

28. $-2x^2$.

29. $\frac{b}{a}$.

30. b.

31. 1.

32. 1.

33. $\frac{3+x^2}{x(1-x^2)}$.

34. $\frac{4b^2+6brx^2}{b^2-r^2x^4}$.

35. $\frac{x(a+b)-ab}{(x-a)(x-b)}$.

36. 0.

37. $\frac{1}{z(x-z)(z-y)}$.

38. -1.

39. 0.

40. 2.

41. 0.

42. 0.

43. -1.

Art. 409.

1. 1.

2. $\frac{abc}{a^2 - ab + b^2}$.

3. 2.

4. $1\frac{1}{2}$.

5. $\frac{ab}{a + b - c}$.

6. $\frac{3}{4}$.

7. 3.

8. $\frac{mn}{m + n}$.

9. $\frac{16}{9}$.

10. 30.

11. $\frac{2ab}{a + b}$.

12. $\frac{a + b}{2}$.

13. $\frac{a + b + 3}{2}$.

14. $\frac{c^2 - ab}{a + b - 2c}$.

15. 2.

16. $\frac{a + b}{2}$.

17. $\frac{ab}{a + b}$.

18. 40.

19. $\frac{bn + c}{n - 1}$.

20. 20.

21. $-\frac{19}{65}$.

22. $\frac{mn}{m + n}$.

23. $2(a + b)$.

Art. 410.

1. 175 lb.

2. 15 cwt.

3. $6\frac{2}{3}$ oz.; $13\frac{1}{3}$ oz.

4. $10\frac{10}{11}$ min.

5. $27\frac{3}{11}$ min.; $43\frac{7}{11}$ min.

6. $16\frac{4}{11}$ min.

7. $54\frac{6}{11}$ min.

8. $27\frac{3}{11}$ min.

9. $32\frac{4}{13}$ min. after 7.

10. $32\frac{4}{13}$ min. after 5.

11. $18\frac{6}{13}$ min. after 8.

12. A, \$1575; B, \$1400.

13. \$300; \$450.

14. \$.70; \$.80.

15. A, 5 mi.; B, 4 mi.

16. $6\frac{2}{3}$ miles.

17. 2 miles per hour.

18. 30 gal.

19. 3 mi.; 12 mi.

20. $18\frac{6}{13}$ min. of 4 o'clock.

21. Thief, 300; officer, 200.

22. \$7400.

23. Watch gains 3 min.; clock loses 1 min.

24. 28 lb.

25. \$280.

Art. 412.

1. 48 yrs.; 16 yrs.
2. \$175; \$35.
3. $3\frac{7}{11}$ days.
4. 2 hours.
5. 3 mi.
6. $27\frac{3}{11}$ min. after 5.
7. 720 mi.; 20 hrs.

Art. 413.

1. Youngest, $\frac{d-6m}{4}$;
   oldest, $\frac{d+6m}{4}$.
2. First, $\frac{ac}{a-b}$;
   second, $\frac{bc}{a-b}$.
3. $\frac{nr-r(r+1)^2}{(r+1)^2}$;
   $\frac{nr+r(r+1)^2}{(r+1)^2}$;
   $\frac{n}{(r+1)^2}$; $\frac{nr^2}{(r+1)^2}$.
4. $\frac{a(m+1)}{mn-1}$; $\frac{a(n+1)}{mn-1}$.
5. $\frac{hmn}{m+n}$ miles.
6. Breadth, $\frac{160d-bc}{ac+b}$ rods;
   length, $\frac{a(160d-bc)}{ac+b}$ rods.
7. $\frac{am-b}{m-1}$ gallons.
8. $\frac{b(d-c)}{a-d}$ pounds.
9. $\frac{ac+d}{b+c}$ days.
10. $\frac{m(d-e-r)}{r}$ men.
11. A, $\frac{a}{4m+3}$;
    B, $\frac{am}{4m+3}$;
    C, $\frac{a(m+1)}{4m+3}$;
    D, $\frac{a(2m+1)}{4m+3}$.
12. A, $\frac{2abc}{ac+bc-ab}$;
    B, $\frac{2abc}{ab+bc-ac}$;
    C, $\frac{2abc}{ab+ac-bc}$;
    all, $\frac{2abc}{ab+ac+bc}$.
13. $\frac{a(c-bd)}{a-b}$.

Art. 414.

1. $x = 5$.
2. $x = 3$.
3. $x = 5$.
4. $x = \frac{16}{15}$.
5. $x = \frac{a}{4}$.
6. $x = \frac{(b-a)^2}{2a-b}$.
7. $x = \frac{(b-a)^2}{2b}$.
8. $x = \frac{4a}{5}$.
9. $x = \frac{9}{26}$.
10. $x = \frac{\sqrt{a}}{\sqrt{a}+2}$.
11. $x = \frac{ab}{1-2\sqrt{b}}$.
12. $x = \frac{1}{b-2}$.
13. $x = a^2 - b$.
14. $x = \frac{2ab-b^2}{2(a-b)}$.
15. $x = \frac{9}{16}a$.
16. $x = \frac{(a-b)^2}{2b}$.
17. $x = \frac{b(b-2a)}{3b-2a}$.
18. $x = \pm\frac{1}{2a}\sqrt{2(b^4-a^4)}$.
19. $x = \pm\frac{a}{2}\sqrt{3}$.
20. $x = \pm\frac{1}{a}\sqrt{a(a+1)}$.
21. $x = \frac{a(n-1)}{\sqrt{2n-1}}$.
22. $x = \pm\frac{2}{5}\sqrt{-6}$.
23. $x = \pm\frac{2a}{1+c}\sqrt{c}$.
24. $x = \pm 2\sqrt{b(a-b)}$.
25. $x = \pm 3$.
26. $x = \frac{-3a \pm a\sqrt{-7}}{8}$.
27. $x = a$, or $-\frac{a}{3}$.
28. $x = \frac{4a}{5}$, or $\frac{3a}{5}$.
29. $x = \frac{a(a \pm \sqrt{a^2-4ab})}{2b} - a$.
30. $x = \frac{b \pm \sqrt{b^2-2ab}}{2}$.
31. $x = \pm\frac{a(b-1)^2}{2b}$.

32. $x = \frac{a(25b^2 - 1)}{25b^2 + 1}$.

33. $x = 8$.

34. 8, or $\frac{1}{25}$.

35. 8, or $-\frac{1}{25}$.

36. 11, or $-\frac{1}{2}$.

37. 11, or -1.

38. $x = \frac{a^2 - bc}{b - c}$.

Art. 415.

1. $x = 1$, or 2.

2. $x = 10$, or -29.

3. $x = 7$, or $\frac{5}{6}$.

4. $x = 44$, or -2.

5. $x = 5$, or -3.

6. $x = 11$, or 3.

7. $x = 2$.

8. $x = 2$, or -2.

9. $x = 3$, or $-\frac{8}{3}$.

10. $x = 6$, or $2\frac{2}{3}$.

11. $x = 8$, or $2\frac{4}{11}$.

12. $x = 3$, or $-4\frac{2}{3}$.

13. $x = 6$, or $2\frac{2}{7}$.

14. $x = 7$, or $2\frac{1}{3}$.

15. $x = 5$, or $-2\frac{1}{2}$.

16. $x = \frac{1}{2}$, or -3.

17. $x = 2$, or -1.

18. $x = 4\frac{1}{3}$, or $\frac{1}{4}$.

19. $x = 21$, or 5.

20. $x = 5$, or $\frac{11}{12}$.

21. $x = \frac{2}{1}\frac{4}{8}$, or $-\frac{5}{3}$.

22. $x = 36$, or 12.

23. $x = 9$, or $-3\frac{1}{3}$.

24. $x = \frac{1}{2}(-5 \pm \sqrt{29})$.

25. $x = \frac{1}{2}(-5 \pm \sqrt{19})$.

26. $x = 6$, or $-9\frac{2}{17}$.

27. $x = 1$, or $-\frac{49}{41}$.

28. $x = 4$, or $\frac{1}{8}$; $y = 8$, or $\frac{1}{4}$.

29. $x = 21$, or -11; $y = 11$, or -21.

30. $x = 5$, or $\frac{1}{2}(-5 \pm \sqrt{-75})$;

$y = 2$, or $-1 \pm \sqrt{-3}$.

31. $x = \pm 5$, or ± 3; $y = \pm 3$, or ± 5.

32. $x = \pm 3$, or ± 4; $y = \pm 4$, or ± 3.

33. $x = 9,\ 6,\ \text{or}\ -9 \mp \sqrt{5}$;
$y = 1,\ 4,\ \text{or}\ -3 \pm \sqrt{5}$.

34. $x = \pm 4$, or $\mp \frac{3}{5}\sqrt{3}$; $y = \pm 5$, or $\pm \frac{16}{3}\sqrt{3}$.

35. $x = \pm 2$, or $\pm \frac{1}{108}$; $y = \pm 3$, or ± 18.

36. $x = \pm 3$, or ± 2; $y = \pm 2$, or ± 3.

Art. 420.

1. $x > 2$.
2. $x > 6$.
3. $x > 6$.
4. $x > 5$.
5. $x > 2d$.
6. $x = 1$.
8. $x > a$.
9. $x < b$.
10. $x = 4$.
11. $x = 5$.
12. 60.
13. 19; 6.

Art. 426.

3. $a - \sqrt{a^2 - b^2}$.
7. $x = -5$, or 2.
8. $x = -3$, or -4.
9. $x = 13$, or -5.
10. $x = -2$, or $-\frac{1}{3}$.
11. $x = -\frac{5}{2}$, or -3.
12. $x = \frac{7}{13}$, or $\frac{1}{3}$.
13. $x = \frac{1}{7}$, or $-\frac{3}{5}$.
14. $x = -1\frac{1}{3}$, or $1\frac{6}{7}$.
15. $x = 2\frac{1}{2}$, or $-\frac{7}{11}$.
16. $x = -1$, or $3\frac{3}{5}$.
17. $x = 3,\ -5$, or 1.
18. $x = -2,\ -3$, or -5.
19. $x = 1,\ 2$, or 7.
20. $x = 1,\ 7$, or -9.
21. $x = -2,\ 3$, or -4.
22. $x = -2,\ 4$, or -8.
23. $x = -2,\ -3$, or 5.
24. $x = -1,\ -5$, or 12.
25. $x = -1,\ 3$, or $-\frac{5}{3}$.
26. $x = 11,\ 2$, or 1.

Art. 433.

1. $x = 3$; $y = 1$.
2. $x = 23$; $y = 2$.

3. $x = 2$; $y = 5$.
4. $x = 15$; $y = 8$.
5. $x = 5$; $y = 6$.
6. $x = 30$; $y = 20$.
7. $x = 30$; $y = 15$.
8. $x = 17$; $y = 13$.
9. $x = 5$; $y = 6$.
10. $x = 5$; $y = 12$.
11. $x = 49$; $y = 22$.
12. $x = 11$; $y = 3$.
13. $x = 29$; $y = 20$.
14. $x = 11$; $y = 8$.
15. $x = 2$; $y = 3$.
16. $x = 2$; $y = 3$.
17. $x = 10$; $y = 2$; $z = 7$.
18. $x = 50$; $y = 40$; $z = 30$.
19. $x = 3$; $y = 4$; $z = 5$.

Art. 434.

1. Calves, 6; hogs, 5.
2. 14, or 10.
3. Hogs, 5; sheep, 7; lambs, 88.
4. 4.
5. 503.
6. 76, 70, 64, or 58.
7. 185 and 15; 119 and 81; 53 and 147.
8. 97.
9. 974.
10. 15 horses; 35 oxen.
11. 6; 11; 13.
12. 10 men; 16 women; 4 children.
13. 381.
14. 25¢, 7; 10¢, 22; 3¢, 8.
15. A, $2685; B, $3759; C, $4833.

Art. 441.

1. 720; 3628800; 40320.
2. 720, 120; 5040, 210; 30240, 252.
3. 479001600.
4. 6435.
5. 5040.
6. 1680; 70.
7. 5200300.
8. 3628800.

Art. 444.

2. $1 + 2x + 3x^2 + 6x^3 + 9x^4 + \text{etc.}$

3. $\frac{1}{2x^2} - \frac{5}{4x} + \frac{15}{8} - \frac{45}{16}x + \frac{135}{32}x^2 - \text{etc.}$

4. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \text{etc.}$

5. $\frac{2}{3x^2} + \frac{2^2}{3^2x} + \frac{2^3}{3^3} + \frac{2^4x}{3^4} + \frac{2^5x^2}{3^5} + \text{etc.}$

Art. 465.

6. When $x = 4$, y increases 12 times as fast as x. When $x = \frac{1}{3}$, y increases at the same rate as x. In general, y increases $3x$ times as fast as x. When x is less than $\frac{1}{3}$, y increases slower than x.

Art. 471.

1. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}.$

2. $1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{5x^4}{8}.$

3. $1 - 2x - x^2 - \frac{4x^3}{3} - \frac{7x^4}{3}.$

4. $1 - 2x + 3x^2 - 4x^3 + 5x^4.$

5. $1 + 6x + 24x^2 + 80x^3 + 240x^4.$

6. $1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \text{etc.}$

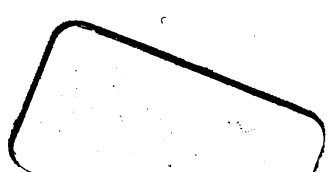
7. $1 + 3x + 9x^2 + 27x^3 + 81x^4 + \text{etc.}$

8. $a^{-1} - 4a^{-5}x + 10a^{-6}x^2 - 20a^{-7}x^3 + 35a^{-8}x^4 - \text{etc.}$

9. $1 - x + \frac{2x^2}{3} - \frac{10x^3}{27} + \frac{5x^4}{27} - \text{etc.}$

10. $1 - \frac{1}{3}x^2 + \frac{1}{9}\frac{4}{1}x^4 - \frac{1}{8}\frac{4}{1}x^6 - \frac{7}{2}\frac{7}{4}\frac{7}{3}x^8 - \text{etc.}$

11. $a^{-4} - 2a^{-6}x^2 + 3a^{-8}x^4 - 4a^{-10}x^6 + \text{etc.}$

12. $a^{\frac{3}{5}} - \frac{1}{5}a^{-\frac{7}{5}}x^2 - \frac{3}{2}\frac{3}{5}a^{-\frac{12}{5}}x^4 - \frac{6}{1}\frac{6}{5}a^{-\frac{2}{5}\frac{2}{5}}x^6 - \frac{2}{6}\frac{2}{2}\frac{1}{5}a^{-\frac{3}{5}\frac{2}{5}}x^8 - \text{etc.}$

13. $a^{\frac{1}{5}} - \frac{4}{5}a^{-\frac{9}{10}}b^{\frac{1}{2}}x - \frac{4}{8}\frac{1}{5}a^{-\frac{1}{5}\frac{3}{5}}bx^2 - \frac{2}{1}\frac{1}{2}\frac{6}{5}a^{-\frac{3}{1}\frac{2}{0}}b^{\frac{3}{2}}x^3 - \text{etc.}$